From a Relativistic Phenomenology of Anyons to a Model of Unification of Forces via the Spencer Theory of Lie Structures

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Abstract

Starting from a relativistic phenomenology of anyons in crystals, we discuss the concept of relativistic interaction and the need to unify electromagnetism and gravitation within the Spencer cohomology of Lie equations. Then, from the sophisticated non-linear Spencer complex of the Poincaré and conformal Lie pseudogroups, we build up a non-linear relative complex assigned to a gauge sequence for electromagnetic and gravitational potentials and fields. Then, using a conformally equivariant Lagrangian density, we deduce, first, the two first steps of its corresponding Janet complex and second, the dual relative linear complex. We conclude by giving suggestions for higher unification with the weak and strong interactions and interpretations of the Lagrangian density as a thermodynamical function and quantum wave-function.

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1. Introduction

In this paper we propose a model as well as suggestions for a unification of physical interactions. This is a model of electromagnetic and gravitational interactions well-founded on a phenomenological relativistic model of anyons in high-T_c superconductors (Rubin 1994). This unification has its roots, first in the Spencer cohomology of the conformal Lie pseudogroup which has abundantly been studied by J. Gasqui & H. Goldschmidt (Gasqui *et al.* 1984), and second, in the non-linear cohomology of Lie equations studied by H. Goldschmidt & D. Spencer (1976a, 1976b, 1978a, 1978b, 1981, see references therein). Meanwhile we only partially refer to some of its aspects to work out a relative non-linear cohomology explicitly associated with a model of unification. Such an approach was originally proposed by J.-F. Pommaret (1988, 1989, 1994) however, it appeared to us to be incomplete, indeed erroneous.

Thus, the purpose of the present paper is to discuss the Pommaret model and to suggest new developments based on the same assumption. Like Pommaret (1989), we think that the geometrical approach of the Maxwell theory has to be modified to be incorporated in a larger theory which explicitly includes gravitation through different equations describing the variations of potentials of gravitation. This result - or proposition - has not been obtained by Pommaret (Pommaret 1994, see conclusions page 456) who could not find any alternative descriptions and justifications neither for the Einstein theory, nor for the equations of fields of gravitation within the frame of the Spencer cohomology.

We conclude this paper a) by succinctly proposing a possible reinterpretation of the quantum wave-function as a classical thermodynamic function within the frame of the Misra-Prigogine-Courbage (MPC) ergodic theory of fields (Misra *et al.* 1979, Misra 1987), b) by suggesting ideas about a unification including the weak and strong interactions along a basic "à la Penrose" approach (Penrose *et al.* 1986).

Furthermore, this work is the result of informal reflexions about an increasing amount of contradictions and incoherences mainly concerning the concept of relativistic interaction, (that we find more and more serious) in the field of quantum physics as well as in classical physics. According to this observation, we first present our motives and a description of these contradictions in relation to F. Lurçat's (1964), J.-M. Lévy-Leblond's (1990) and J.-F. Pommaret's arguments (1989) in order to justify a development via the Spencer cohomology of Lie pseudogroups (Goldschmidt *et al.* 1976a, 1976b, 1978a, 1978b, 1981).

2. Goals and problems

2.1. The physics of crystals and a relativistic phenomenology of anyons

Our initial motivation shall be seen as extremely far from the problems with unifications. Actually, we were more concerned in a simple minor model of a relativistic phenomenology of creation of anyons, accurate for certain crystals (Rubin 1994). At the

origin of this process of creation, we suggested the kinetico-magnetoelectrical effect as described by E. Asher (1973) and which has its roots in the former Minkowski works about the relations between tensors of polarization P and Faraday tensors F in a moving material of optical index $n \neq 1$.These relations are established by turning the following diagram into a commutative one:

$$\begin{array}{ccc} F' & \xrightarrow{\Lambda} & F \\ \downarrow \chi' & & \downarrow \chi \\ P' & \xrightarrow{\Lambda} & P \end{array}$$

where Λ is a Lorentz transformation, allowing us to shift from a frame R' to a frame R, and χ' and χ are respectively the tensors of susceptibility within those two frames, as well supposing P (or P') linearly depending on F (respectively F'). Resulting from this commutativity, the tensor χ linearly depends on χ' in general and also on a velocity 4-vector $\tilde{u}$ associated to Λ (i.e. the relative velocity 4-vector between R and R'). In assimilating R' to the moving crystal frame and R to the laboratory frame, then to an applied electromagnetic field F fixed in R, corresponds in R' a field of polarization P which varies in relation to $\tilde{u}$. This is the so-called kinetico-magnetoelectrical effect.

Parallel to this phenomenon, A. Janner & E. Asher studied the concept of relativistic point symmetry in polarized crystals (Janner *et al.* 1969, 1978). Such a symmetry is defined, on the one hand, by a given discret group G, sub-group of the so-called Shubnikov group $O(3)1'$ associated with the crystal, and on the other hand, as satisfying the following properties: to make this relativistic symmetry exist, there must be a $H(P)$ non-trivial group of Lorentz transformations depending on P, in which G is a normal sub-group, and that leaves the tensor of polarization P invariant. In other words, if $N(G)$ is the normalizer of G in the Lorentz group $O(1,3)$, and $K(P)$ the sub-group of $O(1,3)$ leaving P invariant, then $H(P)$ is the maximal sub-group such that:

$$\left\{ \begin{array}{l} H(P) \subseteq K(P) \cap N(G) \\ H(P) \cap O(3)1' = G. \end{array} \right.$$

We can prove that $H(P)$ is about to exist only if a particular non-vanishing set V of velocity 4-vectors, invariant by action of G, is present and consequently compatible with a kinetico-magnetoelectrical effect (Asher 1973). Therefore, if there is an interaction between moving particles in the crystal and the polarization P, then the trajectories and P are obviously modified, and so is $H(P)$. In this process, only the group $N(G)$ is conserved so that the polarization and the trajectories are deducible during the time by the action of $N(G)$.

As we shall stipulate later on, the existence of an interaction will emerge due to a correlation between the position 3-vectors $\vec{r}$ of the charge carriers and a particular 3 - vector $\vec{w}$ ($\notin V$ in general) associated with P; $\vec{w}$ becoming then a function of $\vec{r}$. In order to allow a cyclotron-type motion which is implicit within the theory of anyons, the group $N(G)$ must contain the group $SO(2)$ and the latter must non-trivially act on all the groups $H(P)$ associated to G. Then, only 12 groups G are compatible with such a description

(Rubin 1993). In fact, throughout this development, we implicitly use a principle of equivalence similar to the one formulated in general relativity: one cannot distinguish a cyclotron - type motion in a constant polarization field from a uniform rectilinear motion in a field of polarization varying in time by action of the normalizer $N(G)$. The time evolution of $\vec{w}$ requiring the explicit knowledge of the gradient $\vec{\nabla}(\vec{w})$ of $\vec{w}$, it follows that $\vec{w}$ and $\vec{\nabla}(\vec{w})$ are respectively the analogues (not the equivalents) of the tetrads and of the Killing vector fields. From an other point of view, the interaction is considered to allow the extension of an invariance with respect to $H(P)$ to an invariance with respect to $N(G)$. The lack of interaction is then what breaks down the symmetry!

This type of reasoning concerns in fact a large amount of physical phenomena such as the spin-orbit interaction for instance. In this context, the cyclotron-type motion of electrons in anyonic states would be similar to the Thomas or Larmor precessions (see also the Coriolis or Einstein-Bass effects). More precisely, taking up again a computation, analogous to the Thomas precession one (Bacry 1967) (i.e. considering as a constant the scalar product of two tangent vectors being two parallel transports along the trajectory (Dieudonné 1971)), concerning a charge carrier with the velocity 4-vector $\tilde{u}$ in R, "polarized" by $\vec{w}(\vec{r})$ such as for example ($\tilde{v} = (0, \vec{v})_R$ constant and $\vec{v} \in V$):

$$\tilde{w} = (0, \vec{w})_R \equiv -P.\tilde{v} \text{ } or \text{ } {}^*P.\tilde{v},$$

where P depends on $\vec{r}$, one can prove from $\tilde{w}.\tilde{u} = cst.$ that (t being the laboratory frame time and ($\tilde{r} = (t, \vec{r})_R$):

$$\frac{d\tilde{u}}{dt} = (-e/m) F_{eff.}(\tilde{r}).\tilde{u}, \tag{1}$$

where m and e are respectively the mass and the electric charge of the carrier and $F_{eff.}(\tilde{r}) \equiv (\vec{E}_{eff.}(\tilde{r}), \vec{B}_{eff.}(\tilde{r}))$ is an effective Faraday tensor such that ($\gamma = (1 - \vec{w}^2)^{-\frac{1}{2}}$ and $\vec{j} = e\vec{u}$):

$$\begin{cases} \vec{B}_{eff.}(\tilde{r}) = (m/e^2) \left(\frac{\gamma}{1+\gamma}\right) \vec{w} \wedge \left[\vec{j}.\vec{\nabla}\right] \vec{w} \\ \vec{E}_{eff.}(\tilde{r}) = \vec{0}. \end{cases}$$

Clearly, $F_{eff.}$ is an element of the Lie algebra of the group $SO(2)$ included in $N(G)$ and with $\vec{B}_{eff.} \in V$. Therefore this $\vec{B}_{eff.}$ magnetic field or $\vec{v}$ (up to a constant) might be considered as the effective magnetic field of the flux-tube V generating the so-called Aharanov-Böhm effect at the origin of the statistical parameter in the anyons theory (Wilczek 1990). In this precession computation, from a more mathematical point of view, taking up the Lie groupoïds theory, we think that perhaps we shift from a source "description" (at $t = 0$) to a target one (at any $t \neq 0$). It is definitely an equivalence principle analogous to the one occuring in general relativity, as shown by J.-F. Pommaret (1989). Let us add that in general $\mathrm{div}(\vec{B}_{eff.}) \neq 0$ so that one gets a non-vanishing density of effective magnetic monopoles generated by the local variations (due to the interaction) of the polarization vector field $\vec{w}$ in the crystal. Thus an anyon would be an effective magnetic monopole associated with a charge carrier, namely a dyon. Moreover, because this effective Faraday tensor is no more a closed two-from, a non-vanishing Chern-Simon

has to be taken into account in a Lagrangian description of anyons, from which non-vanishing spontaneous constant currents can occur.

2.2 Relativistic interaction in quantum mechanics

2.2.a. Polarization in quantum mechanics

The previous vector $\vec{B}_{eff.}$ (or $\vec{v} \in V$) has the same status as the spin. Like the latter, it is defined by a torsor of order 2. From that time on, the transition from a classical description to a quantum one means that one must give an account of the interaction between a free particle (constituting a first sub-system) and a torsor field of order 2 (constituting a second sub-system). The problem seems to be solved and in particular the spin appears to be a "minor" complication of the wave-function defined on the Minkowski space-time, i.e. on a "non-polarized" space. This has to be taken as a postulate, an erroneous one according to part of F. Lurçat's (1964) and J.-M. Lévy-Leblond's (1990) arguments.

In fact, it is not even the case according to classical Galilean mechanics. Considering a body at a given time, one needs to know 6 parameters to describe it: 3 for the position and 3 others for the orientation. The latter are forgotten during the transition from the "extended body to the punctual particle" according to the quantum description. This can only be justified providing that the energy of rotation is negligible compared to the energy of translation. This is the indicator of an inadequacy of the principle of correspondence from classical mechanics to quantum mechanics. But the transition from quantum to classical mechanics is equally problematic: reaching the limit $\hbar \to 0$, the spin vanishes. Finally, we must say that this correspondance does not exist any more in chromodynamics.

Therefore, the fact that the wave-function only depends on the position should rather be considered as a postulate (moreover, the fact is inexistant in classical mechanics). For instance, if we measure the electric and magnetic fields at a point in space-time with a system of coordinates defined by a given Lorentz transformation, we can deduce that the Faraday tensor is a function of the 4 parameters of the position and 6 others defining Λ. It explicitly occurs within the kinetico-magnetoelectric effect.

Refering now to the wave-equations and to the methods usually accepted to determine them, we then use a principle of invariance. First, we identify the appropriate space-time symmetries, that is the group of relativity of the theory (for example the Poincaré group). Then, according to Wigner's theory, we build the irreducible unitary representations (eventually projective) of the group to which correspond the elementary "kinetic" objects of the theory (i.e. the vector-valued wave-functions). Then again, we derive the infinitesimal generators of the group that we identify with physical and geometrical observables like the energy, the kinetic moment, etc...; the Lie algebra of the group and its "quantum" extensions defining the commutators. Finally, determining the invariants through the action of the group we obtain the other physical observables and their commutators.

Unfortunately, some ambiguities appear. For example, the generalization of the Dirac theory for spins higher than 1/2 gives different non-equivalent possible wave-equations.

Lastly, if we consider a theory of interacting fields, the particles associated with these fields can get out of their mass shell but not out of their spin shell! We then forget the spin again, which is impossible within a "(m,s)" theory. Therefore some of the aspects of the interaction and of the wave-equations - to be brief- can fully account for neither the free particles, nor the interacting particles!

In order to escape from these contradictions and to best describe the free particles, F. Lurçat proposed an approach "à la Wigner". In this way, he postulated that first the scalar complex wave-function of a free particle was defined on the "Poincaré space" of the Poincaré Lie group. Second, this wave-function is an eigenfunction of the two Casimirs of the Poincaré group. Thus, the wave-function ϕ is a function first, of a 4-vector position $\tilde{x}$, element of the dual Lie algebra of the group of translations and second, of a second order torsor F (like the Faraday torsor which is an element of the dual Lie algebra of the Lorentz group). Also in fact, the Poincaré Lie group on the "Poincaré space" is the action of a so-called Lie groupoïd on its associated Lie algebroïd, defined on the Minkowski space-time. Finally, if we want to describe the interaction, making these modifications and requiring a gauge invariance, the involution of the infinitesimal generators "deformed" by the gauge fields is no longer satisfied, nor are the relativistic invariance and the correspondence principle.

This breaking of the invariance can be seen with the Dirac equation. More exactly, the Dirac equation is not *equivariant*, but only covariant, meaning that this equation is not invariant under any conformal changes of coordinates, but only under a change of frame, i.e. a change of basis of the tangent Minkowski space. In that case, for instance, one of the spectacular manifestations is Klein's paradox of non-conservation of the current of probability during the scattering in a square potential (Itzykson *et al.* 1980). Still, the idea stays that ϕ is defined on a "larger" space-time than the Minkowski one, but for being a Kaluza-Klein type theory for example.

2.2.b. Concerning the contradictions of an approach "à la Wigner" and the Einstein-Cartan unification

Let us consider a complex wave-function ϕ depending on $\tilde{x}$ and F on which the Poincaré group projectively acts. Because, then, of the phase arbitrariness, the ten infinitesimal generators of the group are defined with ten arbitrary gauge potentials, so-called Poincaré potentials. In order to keep the Lie algebra structure, i.e. the involution of the deformed generators, the fields associated with these potentials must satisfy some constraint equations. The electromagnetic field associated with the translations of the group has especially to be vanishing, which is first absolutly contradictory with ϕ as a function of F. Second, that would mean that one cannot describe any interaction with a field F without a Poincaré symmetry breaking. Nevertheless this symmetry is necessary to keep the relativistic equivariance.

Moreover, ϕ is independent of the gauge potentials although ϕ should be parametrized by these functions refering to the formal theory of system of partial differential equations (PDE). In fact, according to the Kähler-Cartan theorem, the analytical solutions of an involutive system of PDE (so formally integrable) depend on certain constants of integra-

tion (Dieudonné 1971), but, contrary to the ODE (ordinary differential equations), they depend on arbitrary C^∞ functions only constrained to verify Cauchy initial data (see also Shih 1986, 1987, 1991). These functions should be identified with the gauge potentials and with their fields. Still, we can say that the incoherences are not over!

Let us assume the presence of a relativistic interaction, then there consequently exists, if we refer to the equivalence principle, a proper frame in which ϕ is stationnary. If τ is the "proper" time then $\partial_\tau \phi = 0$. Within the "laboratory" frame, we will get a different equation, moreover, there won't be any relativistic equivariance; a problem that is similarly encountered with either the Newton-Wigner position operator or the Dirac position operator. Indeed we will still be unable to write down this equation (!) for the reason that to shift to the Laboratory frame, one needs to know the classical motion of the particle, i.e. to know the equation that determines the evolution of the 4-vector speed $\tilde{u}$ identified with the basic time type 4-vector $\tilde{e}_0$ of the proper frame.

Also at this point, the equation should be established in a certain system of coordinates and from $\tilde{e}_0$ only, one should obtain the other basic vectors of space type $\tilde{e}_i$ (i=1,2,3) as well as their time evolutions. This equation should then be integrable in the Fröbenius sense. We think it is perhaps a matter of a generalisation of the Frênet moving frame method, as formulated by E. Cartan. Whereas we observe that by determining this moving frame from $\tilde{e}_0$ and $\dot{\tilde{e}}_0$, that the $\dot{\tilde{e}}_i$'s are defined from the $\tilde{e}_\mu$'s (μ=0,1,2,3) and from the third order time derivatives of $\tilde{e}_0$. Then, if $\dot{\tilde{e}}_0 \equiv F.\tilde{e}_0$, we need to know the time derivatives of F up to the second order. Therefore, in the laboratory frame, ϕ should depend on the derivatives of F, contrary to the initial assumption, unless these derivatives are themselves functions of F and $\tilde{x}$. Condition which, first, is not the case in the Maxwell theory, second would set down some constraints on the moving frame and consequently a "partial" equivariance; unless one completes it with other fields - like those associated for example with the space-time curvature - which would be related to the derivatives of F. Let us remark also that this kind of discussion seems to be very similar to the one encountered in the demonstration of the Cauchy-Kowalewska theorem when starting from a pfaffian system to a "normal system" of PDE's and considering the so-called "regular" tangent spaces and integrable manifolds (Dieudonné 1971).

On that subject, one can notice that the equation (1) can be rewritten in an orthonormal system of local coordinates ($\alpha, \beta, \gamma = 1, ..., n$; the Γ's being the Christoffel symbols):

$$\dot{u}^\alpha + \Gamma^\alpha_{\beta,\gamma} u^\beta u^\gamma = 0.$$

We recognize the equation of the geodesics associated to a Riemannian connexion with torsion. This would suggest a unification in reference to the Einstein-Cartan theory. If we then keep on with the assumption that one has to add to the electromagnetic field a gravitational field and that the derivatives of the fields are functions of the fields themselves (as with the Bianchi identities according to the non-abelian theory for example), that means we make the assumption of the existence of a differential sequence. In electromagnetism, it is a matter of the de Rham sequence but gravitation does not interfere. The sequence integrating the latter - and being the purpose of this paper - might be a certain generalizing complex like the Spencer one, following then a method proposed by J.-F. Pommaret (1994)but largely modified.

3. The Lie conformal pseudogroup associated to the unification model

First of all, let us assume that the group of relativity is not the Poincaré group anymore but the conformal Lie group (we know from Bateman and Cunningham studies (1910) that it is the group of invariance of the Maxwell equations). In particular, this involves that no changes occur shifting from a given frame to a uniformly accelerated relative one. From a historical point of view, that happened to be the starting point of the Weyl theory which was finally in contradiction with experimental data and for various other reasons presented, for instance by J.-F. Pommaret in the framework of the Janet and Spencer complexes (Pommaret 1989). Then, starting from this mathematical framework, J.-F. Pommaret considered in trying this unification, the linear Spencer complex defined from the system of finite Lie equations associated to the Lie pseudogroup of conformal isometries. Unfortunatly, the system of PDE proposed by J.-F. Pommaret is incomplete and its conditions of use are not really given. On the other hand, he claimed the Spencer complex of the conformal Lie pseudogroup would be the "unification complex" (Pommaret 1988), whereas we merely prove that it would rather be a relative complex deduced from an abelian extension (Godschmidt 1976a, 1976b, 1978a, 1978b, 1981), when reaching the conformal Lie pseudogroup starting from the Poincaré one. Before tackling these various complexes, we present and recall a few relations concerning the conformal Lie pseudogroup action on some tensors such as the metric and the Riemann and Weyl curvatures. Let us first call $\mathcal{M}$, the base space (or space-time), assumed to be of class C^∞, of dimension $n \geq 4$, connected, paracompact, without boundaries, oriented and endowed with a metric 2-form ω, symmetric, of class C^2 on $\mathcal{M}$ and non-degenerated. We also assume $\mathcal{M}$ to be conformally flat.

3.1. The conformal finite Lie equations

These equations are deduced from the conformal action on the metric. Let us consider $\hat{f} \in Diff^1_{loc.}(\mathcal{M})$ and any $\alpha \in C^0(\mathcal{M}, \mathbb{R})$, then if $\hat{f} \in \Gamma_{\widehat{G}}$ ($\Gamma_{\widehat{G}}$ being the pseudogroup of local conformal bidifferential maps on $\mathcal{M}$), $\hat{f}$ is a solution of the following system of PDE (other PDE must be satisfied to completly define $\Gamma_{\widehat{G}}$):

$$\begin{cases} \hat{f}^*\omega = e^{2\alpha}\omega \\ \text{and with } \det(J(\hat{f})) \neq 0, \end{cases} \tag{2}$$

where $J(\hat{f})$ is the Jacobian of $\hat{f}$, and $\hat{f}^*$ is the pull-back of $\hat{f}$. We denote $\tilde{\omega}$ the metric on $\mathcal{M}$ such as:

$$\tilde{\omega} \overset{def.}{\equiv} e^{2\alpha}\omega,$$

and we agree to put a tilde on each tensor or geometrical "object" relative to or deduced from this metric $\tilde{\omega}$. Let us notice that the latter depends on a fixed given element $\hat{f}^* \in \Gamma_{\widehat{G}}$. Also, in order to properly recall the last point, we shall sometimes use an alternative notation such as:

$$\tilde{\omega} \stackrel{def.}{\equiv} {}^{\hat{f}}\omega.$$

This convention of notation will also be used on each geomerical object relative to this metric. Now, doing a first prolongation of the system (2), we deduce other second order PDE connecting the affine connexion 1-forms of Levi-Civita ∇ and $\widetilde{\nabla}$ respectively associated to ω and $\tilde{\omega}$. To obtain these PDE, we merely start from the following definition of $\widetilde{\nabla}$: let X, Y and Z be any vector fields in $C^1(T\mathcal{M})$, $\hat{f} \in Diff^2_{loc.}(\mathcal{M})$ and $\alpha \in C^1(\mathcal{M}, \mathbb{R})$, then by definition we have:

$$\begin{aligned} \tilde{\omega}(\tilde{\nabla}_X Y, Z) &= \tfrac{1}{2}\{\tilde{\omega}([X,Y],Z) + \tilde{\omega}([Z,X],Y) + \tilde{\omega}([Z,Y],Y) \\ &\quad + X.\tilde{\omega}(Y,Z) + Y.\tilde{\omega}(X,Z) - Z.\tilde{\omega}(X,Y)\}, \end{aligned} \tag{3}$$

from which we deduce with the relation (2) $\forall\, X, Y \in C^1(T\mathcal{M})$ and $\forall \hat{f} \in Diff^2_{loc.}(\mathcal{M})$,

$$\widetilde{\nabla}_X Y = \nabla_X Y + \mathbf{d}\alpha(X)Y + \mathbf{d}\alpha(Y)X - \omega(X,Y)\,{}_*\mathbf{d}\alpha, \tag{4}$$

where $\mathbf{d}$ is the exterior differential and ${}_*\mathbf{d}\alpha$ is the dual vector field of the 1-form $\mathbf{d}\alpha$ with respect to the metric ω, i.e. such as $\forall\, X \in T\mathcal{M}$

$$\omega(X, {}_*\mathbf{d}\alpha) = \mathbf{d}\alpha(X) =< \mathbf{d}\alpha | X > . \tag{5}$$

Let us also agree to denote in the sequel $\forall\, \tilde{x} \in \mathcal{M}$, $\forall\, \nu \in \bigwedge^r T^*\mathcal{M}$ and $\forall\, \xi_i \in T\mathcal{M}$ $(i = 1, ..., r)$:

$$\nu_{\tilde{x}}(\xi_{1,\tilde{x}}, ..., \xi_{r,\tilde{x}}) =< \nu(\tilde{x}) | \xi_{1,\tilde{x}} \otimes ... \otimes \xi_{r,\tilde{x}} > .$$

Prolonging again and using the definition of the Riemann tensor ρ associated to ω, i.e. $\forall\, X, Y \in C^2(T\mathcal{M})$

$$\rho(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]},$$

one obtains the following relation $\forall\, X, Y, Z \in C^2(T\mathcal{M})$, $\forall\, \alpha \in C^2(\mathcal{M}, \mathbb{R})$ and $\forall\, \hat{f} \in Diff^3_{loc.}(\mathcal{M})$,

$$\begin{aligned} \tilde{\rho}(X,Y).Z &= \rho(X,Y).Z + \omega(X,Z)\nabla_Y({}_*\mathbf{d}\alpha) - \omega(Y,Z)\nabla_X({}_*\mathbf{d}\alpha) \\ &\quad + \{\omega(\nabla_X({}_*\mathbf{d}\alpha), Z) + \omega(X,Z)\mathbf{d}\alpha({}_*\mathbf{d}\alpha)\}\, Y \\ &\quad - \{\omega(\nabla_Y({}_*\mathbf{d}\alpha), Z) + \omega(Y,Z)\mathbf{d}\alpha({}_*\mathbf{d}\alpha)\}\, X \\ &\quad + \{\mathbf{d}\alpha(X)\omega(Y,Z) - \mathbf{d}\alpha(Y)\omega(X,Z)\}\, {}_*\mathbf{d}\alpha \\ &\quad + \{\mathbf{d}\alpha(Y)X - \mathbf{d}\alpha(X)Y\}\, \mathbf{d}\alpha(Z) \end{aligned} \tag{6}$$

Assuming $\mathcal{M}$ to be comformally flat, the Weyl tensor τ associated with ω vanishes. Hence, the Riemann tensor ρ can be rewritten $\forall\, X, Y, Z, U \in C^2(T\mathcal{M})$ as:

$$\begin{aligned} \omega(U, \rho(X,Y).Z) &= \tfrac{1}{(n-2)}\, \{\omega(X,U)\sigma(Y,Z) - \omega(Y,U)\sigma(X,Z) \\ &\quad + \omega(Y,Z)\sigma(X,U) - \omega(X,Z)\sigma(Y,U)\}\,, \end{aligned} \tag{7}$$

where σ is the so-called Schouten tensor (Gasqui /it et al. 1984) $\forall\, X, Y \in C^2(T\mathcal{M})$,

$$\sigma(X,Y) = \rho_{ic}(X,Y) - \frac{\rho_s}{2(n-1)}\omega(X,Y), \tag{8}$$

where ρ_{ic} is the Ricci tensor and ρ_s is the Riemann scalar curvature. Thus, we might consider the relation (7) as the existence of a short exact sequence "symbolically" written as "$0 \to \sigma \to \rho \to \tau \to 0$" and perhaps related to a sequence of cohomology spaces of symbols (Gasqui *et al.* 1984, Pommaret 1994). Consequently, the system of PDE (6) can be rewritten as a first order system of PDE concerning σ. To do this, we first define two suitable trace operators, used in the sequel to obtain the $\tilde{\rho}_{ic}$ and $\tilde{\rho}_s$ tensors and finally the $\tilde{\sigma}$ tensor. Let us denote Tr^1 the trace operator defined such that for any vector bundle E over $\mathcal{M}$ we have:

$$Tr^1 : T\mathcal{M} \otimes T^*\mathcal{M} \otimes E \longrightarrow E,$$

with

$$Tr^1(X \otimes \alpha \otimes \mu) = \alpha(X)\mu$$

for any $X \in T\mathcal{M}$, $\alpha \in T^*\mathcal{M}$ and $\mu \in E$. Then, the second trace operator is the natural trace Tr_ω associated to ω and defined by:

$$Tr_\omega : \overset{2}{\otimes} T^*\mathcal{M} \longrightarrow \mathbb{R},$$

such that

$$Tr_\omega(u \otimes v) = v({}_*u).$$

Finally, with $Tr^1\tilde{\rho} = \tilde{\rho}_{ic}$ and $Tr_\omega\tilde{\rho}_{ic} = \tilde{\rho}_s$, we deduce first from the relations (6) and (8) $\forall\, \hat{f} \in Diff^3_{loc.}(\mathcal{M})$, $\forall\, X, Y \in C^2(T\mathcal{M})$ and $\forall\, \alpha \in C^2(\mathcal{M}, \mathbb{R})$,

$$\begin{aligned} \tilde{\sigma}(X,Y) \;\; = \;\; & \sigma(X,Y) + (n-2)\big(\mathbf{d}\alpha(X)\mathbf{d}\alpha(Y) - \omega(\nabla_X({}_*\mathbf{d}\alpha), Y) \\ & -\tfrac{1}{2}\omega(X,Y)\mathbf{d}\alpha({}_*\mathbf{d}\alpha)\big)\,. \end{aligned} \tag{9}$$

In fact this expression can be symmetrized. Indeed, from the following property satisfied by ∇:

$$\omega(\nabla_X({}_*\mathbf{d}\alpha), Y) + \omega({}_*\mathbf{d}\alpha, \nabla_X Y) = X.\,\omega({}_*\mathbf{d}\alpha, Y),$$

and the definition of ${}_*\mathbf{d}\alpha$, one obtains:

$$\omega(\nabla_X({}_*\mathbf{d}\alpha), Y) = X.\,\mathbf{d}\alpha(Y) - \mathbf{d}\alpha(\nabla_X Y).$$

But, from the torsion free property of ∇ and from the relation

$$\mathbf{d}\alpha([X,Y]) = X.\,\mathbf{d}\alpha(Y) - Y.\,\mathbf{d}\alpha(X),$$

one deduces:

$$\omega(\nabla_X({}_*\mathbf{d}\alpha), Y) = Y.\mathbf{d}\alpha(X) - \mathbf{d}\alpha(\nabla_Y X).$$

Now, $\forall\, \alpha \in C^2(\mathcal{M}, \mathbb{R})$, $\forall\, X, Y \in C^1(T\mathcal{M})$, defining $\mu \in C^0(S_2 T^*\mathcal{M})$ by:

$$\mu(X,Y) = \frac{1}{2}\left[X.\,\mathbf{d}\alpha(Y) + Y.\,\mathbf{d}\alpha(X)\right], \tag{10}$$

one has the relation:

$$\omega(\nabla_X({}_*\mathbf{d}\alpha), Y) = \mu(X,Y) - \frac{1}{2}\mathbf{d}\alpha(\nabla_X Y + \nabla_Y X).$$

Then, we can rewrite the first order PDE (9), $\forall\, \hat{f} \in Diff^3_{loc.}(\mathcal{M})$, $\forall\, X, Y \in C^2(T\mathcal{M})$ and $\forall\, \alpha \in C^2(\mathcal{M}, \mathbb{R})$ as:

$$\begin{array}{rcl} \tilde{\sigma}(X,Y) & = & \sigma(X,Y) + (n-2)\,\big(\mathbf{d}\alpha(X)\mathbf{d}\alpha(Y) \\ & & -\mu(X,Y) + \frac{1}{2}\mathbf{d}\alpha(\nabla_X Y + \nabla_Y X) \\ & & -\frac{1}{2}\omega(X,Y)\mathbf{d}\alpha({}_*\mathbf{d}\alpha)\big)\,. \end{array} \tag{11}$$

It is worthy of note that if $\rho_s = c_0$ ($c_0 \in \mathbb{R}$), then the Schouten tensor σ satisfies the relation:

$$\sigma = c_0 \frac{(n-2)}{2}\omega. \tag{12}$$

and only in this case, the equation (11) must become the equation 2 ans so it desappears. Then, considering the system (2), the system (11) reduces to a second order system of PDE such as $\forall\, \alpha \in C^2(\mathcal{M}, \mathbb{R})$ and $\forall\, X, Y \in C^1(T\mathcal{M})$ we have:

$$\begin{array}{rcl} \mu(X,Y) & = & \frac{1}{2}\left\{\left[c_0\,(1-e^{2\alpha}) - \mathbf{d}\alpha({}_*\mathbf{d}\alpha)\right]\omega(X,Y) + \mathbf{d}\alpha\,(\nabla_X Y + \nabla_Y X)\right\} \\ & & -\mathbf{d}\alpha(X)\mathbf{d}\alpha(Y). \end{array} \tag{13}$$

We also have the following PDE deduced from (4), $\forall\, X \in C^1(T\mathcal{M})$ and $\forall\, \hat{f} \in Diff^2_{loc.}(\mathcal{M})$:

$$Tr^1(\widetilde{\nabla}_X) = Tr^1(\nabla_X) + n\,\mathbf{d}\alpha(X). \tag{14}$$

Thus, we have a serie of PDE deduced from (2), in particular made of the systems of PDE (2), (11) and (14). But there are alternative versions of these PDE in which the function $\alpha \in C^2(\mathcal{M}, \mathbb{R})$ doesn't appear. These latters are the following: from the system (2), one deduces, $\forall\, \hat{f} \in Diff^1_{loc.}(\mathcal{M})$:

$$\left\{ \begin{array}{l} \hat{f}^*\hat{\omega} = \hat{\omega} \\ \text{and with } \det(J(\hat{f})) \neq 0, \end{array} \right. \tag{15}$$

where $\hat{\omega} = \omega / |\det(\omega)|^{1/n}$. Then by prolongation, with $\widehat{\nabla}$ and $\hat{\rho}$ being respectively the connexion of Levi-Civita and the Riemann curvature tensor associated to $\hat{\omega}$, one obtains $\forall\, \hat{f} \in Diff^3_{loc.}(\mathcal{M})$:

$$^{\hat{f}}\widehat{\nabla} = \widehat{\nabla} \tag{16}$$

and

$$^{\hat{f}}\hat{\rho} = \hat{\rho}. \tag{17}$$

In the latter system (17) of PDE (first order), it has to be noted that $\hat{\rho} = \tau$, i.e. $\hat{\rho}$ is the Weyl curvature tensor associated to the metric ω. Furthermore, with the assumption that the conformal structure is flat, one has $\hat{\rho} = 0$ and $\hat{\sigma} = 0$. But in general, it is noteworthy to add that if $\tau = 0$ then obviously σ doesn't vanish. Then, the conformal Lie pseudogroup $\Gamma_{\widehat{G}}$ is the set of functions $\hat{f} \in Diff^3_{loc.}(\mathcal{M})$ satisfying the following involutive system of PDE:

$$\begin{cases} \hat{f}^*\hat{\omega} = \hat{\omega} & \text{and with} \qquad \det(J(\hat{f})) \neq 0 \\ ^{\hat{f}}\widehat{\nabla} = \widehat{\nabla}, & \end{cases} \tag{18}$$

completed with a third system of PDE of order 3 defined $\forall\, X, Y \in C^2(T\mathcal{M})$ by:

$$^{\hat{f}}\widehat{\nabla}_X {}^{\hat{f}}\widehat{\nabla}_Y = \widehat{\nabla}_X \widehat{\nabla}_Y. \tag{19}$$

This system is formally integrable if and only if $\tau = 0$ (from the Weyl theorem) and involutive because the corresponding symbol $\widehat{M}_3$ vanishes. From a terminological point of view, one shall say that the system (18)-(19) is the "Lie form" of the system made of the PDE (2), (11) and (14) to which one adds the third order system deduced from the expression of $^{\hat{f}}\widehat{\nabla}_X {}^{\hat{f}}\widehat{\nabla}_Y$ with respect to $\nabla_X \nabla_Y$ and $\alpha \in C^3(\mathcal{M}, \mathbb{R})$. We shall call this latter system equivalent to the system (18)-(19), the "deformed" or "extended" system. It is remarkable that the deformed system brings out a supplementary system in comparison with its Lie form (18)-(19). It is about the system (11). If we consider the second order sub-system deduced from the system (18)-(19), we notice it is still formally integrable (again because of the Weyl theorem) but it is no longer involutive because the symbol $\widehat{M}_2$ of (18)-(19) is only 2-acyclic.

3.2. The conformal Lie groupoïd

Before defining this groupoïd, we need to recall some definitions concerning the sheafs of the k-jet fiber bundles (Kumpera *et al.* 1972). First of all, we denote $J_k(\mathcal{M})$ the affine fiber bundle of the k-jets of local $C^\infty(\mathcal{M}, \mathcal{M})$ functions on $\mathcal{M}$, $\theta_{\mathcal{M}}$ (or simply θ) the sheaf of local rings of germs of continuous functions on $\mathcal{M}$ with values in $\mathbb{R}$, and J_k the affine fiber bundle of the k-jets of local functions in $C^\infty(\mathcal{M}, \mathbb{R})$. Then in what follows, we agree to underline all the names used for the sheafs of germs of local continuous sections associated with the various fiber bundles. Now, if ϵ is a sheaf of θ-modules on $\mathcal{M}$, we conventionally define the sheaf $\underline{J_k(\epsilon)}$ as:

$$\underline{J_k(\epsilon)} \equiv \underline{J_k} \otimes_\theta \epsilon.$$

Then, we have the injective sheafs map:

$$\begin{array}{rccc} j_k: & \underline{C^\infty(\mathcal{M})} & \longrightarrow & \underline{J_k(\mathcal{M})} \\ & & & \\ & [m \to f(m)]_x & \longrightarrow & [m \to j_k(f)(m)]_x \equiv j_k([f])_x, \end{array}$$

where $j_k(f)(m)$ is the set of germs at m of the derivatives of f up to order k, and $[\ ]_x$ obviously being the equivalence classes of local sections at $x \in \mathcal{M}$. Let us also denote $Aut(\mathcal{M})$ the sheaf of germs of functions $f \in Diff^\infty_{loc.}(\mathcal{M})$. The source map such as:

$$\begin{array}{rccc} \alpha_k: & J_k(\mathcal{M}) & \longrightarrow & \mathcal{M} \\ & j_k(f)(x) & \longrightarrow & x, \end{array}$$

and the target map:

$$\begin{array}{rccc} \beta_k: & J_k(\mathcal{M}) & \longrightarrow & \mathcal{M} \\ & j_k(f)(x) & \longrightarrow & f(x), \end{array}$$

are submersions on $\mathcal{M}$. One defines the composition on $J_k(\mathcal{M})$ by:

$$J_k(g)(y) \,.\, J_k(f)(x) = J_k(g \circ f)(x),$$

with $y = f(x)$. The units in $\underline{J_k(\mathcal{M})}$ are the elements $j_k([id])_x$ and they can be identified with the points $x \in \mathcal{M}$. Then, let $\Pi_k(\mathcal{M})$ be the Lie groupoïd of invertible elements of $J_k(\mathcal{M})$. The elements of $\Pi_k(\mathcal{M})$ are the k-jets of the functions $f \in Diff^\infty_{loc.}$. $\underline{J_k(\mathcal{M})}$ (resp.$\underline{\Pi_k(\mathcal{M})}$) is also the sheaf of germs of local continuous sections f_k of α_k (resp. $\alpha_k/\Pi_k(\mathcal{M})$). The sheaf map j_k can be also restricted to the sheaf map:

$$j_k: Aut(\mathcal{M}) \longrightarrow \underline{\Pi_k(\mathcal{M})}.$$

An element $[f_k] \in \underline{\Pi_k(\mathcal{M})}$ shall be called "admissible" if $f = \beta_k \circ f_k \in Aut(\mathcal{M})$ (i.e. $\det([j_1(f)]) \neq 0$). The admissible elements are the germs of continuous sections f_k of $\alpha_k: \Pi_k(\mathcal{M}) \to \mathcal{M}$ such as $\beta_k \circ f_k \in Diff^\infty_{loc.}(\mathcal{M})$. We denote $\Gamma_k(\mathcal{M})$ the sub-sheaf of admissible elements of $\underline{\Pi_k(\mathcal{M})}$. Then, we can define the sheaf epimorphism of groupoïds:

$$j_k: Aut(\mathcal{M}) \longrightarrow \Gamma_k(\mathcal{M}).$$

Finally, we define the source map a_k and the target map b_k in $\underline{J_k(\mathcal{M})}$ by:

$$\begin{array}{rccc} a_k: & \underline{J_k(\mathcal{M})} & \longrightarrow & \mathcal{M} \\ & [\sigma_k]_x & \longrightarrow & x, \end{array}$$

$$\begin{array}{rccc} b_k: & \underline{J_k(\mathcal{M})} & \longrightarrow & \mathcal{M} \\ & [\sigma_k]_x & \longrightarrow & \beta_k \circ \sigma_k(x), \end{array}$$

and the canonical projection Π^p_q $(p \geq q)$ by:

$$\begin{array}{rccc} \Pi^p_q: & \underline{J_p(\mathcal{M})} & \longrightarrow & \underline{J_q(\mathcal{M})} \\ & [f_p]_x & \longrightarrow & [f_q]_x. \end{array}$$

Thus, at the sheafs level, the non-linear finite Lie groupoïd $\widehat{\mathcal{R}}_3$ of the conformal pseudogroup $\Gamma_{\widehat{G}}$ is the set of germs of continuous sections in $\Gamma_3(\mathcal{M})$ satisfying the algebraic equations over each point $x \in \mathcal{M}$, obtained by substituting the germs $[\hat{f}_3]_x \in \Gamma_3(\mathcal{M})$ for the derivatives of $\hat{f}$ up to order three in the system of PDE (18)-(19). In other terms, we substitute $[\hat{f}_3]$ for $j_3([\hat{f}])$. More precisely, one factorizes each differential operator of the system (18)-(19) with the operators j_k $(k = 1, 2, 3)$. Then, one defines the morphisms:

$$\left\{ \begin{array}{rcll} M(\hat{\omega}): & j_1(\mathcal{M}) & \longrightarrow & S_2T^*\mathcal{M} \\ L(j_1(\hat{\omega})): & j_2(\mathcal{M}) & \longrightarrow & T\mathcal{M} \otimes T^*\mathcal{M} \otimes J_1^*(T\mathcal{M}) \\ K(j_2(\hat{\omega})): & j_3(\mathcal{M}) & \longrightarrow & T\mathcal{M} \otimes T^*\mathcal{M} \otimes J_1^*(T\mathcal{M}) \otimes J_2^*(T\mathcal{M}) \end{array} \right.$$

by the respective following relations:

$$\left\{ \begin{array}{rcl} \hat{f}^*\hat{\omega} & = & M(\hat{\omega}) \circ j_1(\hat{f}) \\ {}^{\hat{f}}\widehat{\nabla} & = & L(j_1(\hat{\omega})) \circ j_2(\hat{f}) \\ {}^{\hat{f}}\widehat{\nabla}{}^{\hat{f}}\widehat{\nabla} & = & K(j_2(\hat{\omega})) \circ j_3(\hat{f}), \end{array} \right.$$

and consequently $\widehat{\mathcal{R}}_3$ can be rewritten as the system of PDE made of the two systems of PDE $\widehat{\mathcal{R}}_2$:

$$\left\{ \begin{array}{rcl} {}^{[\hat{f}_1]}\hat{\omega} & \stackrel{def.}{\equiv} & M(\hat{\omega})([\hat{f}_1]) = \hat{\omega} \text{ with } \det([\hat{f}_1]) \neq 0 \text{ and } \det([j_1(\hat{f})]) \neq 0 \\ {}^{[\hat{f}_2]}\widehat{\nabla} & \stackrel{def.}{\equiv} & L(j_1(\hat{\omega}))([\hat{f}_2]) = \widehat{\nabla}, \end{array} \right. \tag{20}$$

completed with the third system:

$${}^{[\hat{f}_3]}(\widehat{\nabla}\widehat{\nabla}) \stackrel{def.}{\equiv} K(j_2(\hat{\omega}))([\hat{f}_3]) = \widehat{\nabla}\widehat{\nabla}. \tag{21}$$

$\Gamma_{\widehat{G}}$ is then the set of germs $[\hat{f}] \in Aut(\mathcal{M})$ such as $j_3([\hat{f}]) \in \widehat{\mathcal{R}}_3$. Now since $\widehat{M}_3 = 0$, one also has the equivalence $\widehat{\mathcal{R}}_3 \simeq \widehat{\mathcal{R}}_2$ and in order to work out the sophisticated non-linear Spencer complex of $\widehat{\mathcal{R}}_3$, it is sufficient, as we shall see later, to obtain it for $\widehat{\mathcal{R}}_2$. That is because the exactness of this complex of length two, only needs the symbol of the corresponding Lie groupoïd to be 2-acyclic. It is precisely the case for $\widehat{M}_2$. Thus, the discussion in what follows will concern exclusively $\widehat{\mathcal{R}}_2$ defined by the "deformed" or "extended" system: $\forall\, X, Y \in C^2(T\mathcal{M})$ and $\forall\, [\alpha_2] = ([\alpha], [\beta], [\mu]) \in \underline{J_2}$,

$$\widehat{\mathcal{R}}_2: \left\{ \begin{array}{rcl} {}^{[\hat{f}_1]}\omega & = & e^{2[\alpha]}\omega \text{ with } \det([\hat{f}_1]) \neq 0 \text{ and } \det([j_1(\hat{f})]) \neq 0 \\ {}^{[\hat{f}_2]}\nabla_X Y & = & \nabla_X Y + [\beta](X)Y + [\beta](Y)X - \omega(X,Y)\,{}_*[\beta] \\ {}^{[\hat{f}_1]}\sigma(X,Y) & = & \sigma(X,Y) + (n-2)\Big([\beta](X)[\beta](Y) - \frac{1}{2}\omega(X,Y)[\beta]({}_*[\beta]) \\ & & \quad - [\mu](X,Y) + \frac{1}{2}[\beta](\nabla_X Y + \nabla_Y X)\Big). \end{array} \right. \tag{22}$$

Let us precise again that the equation (11) makes sense, and in the formula (22) the last equation must be considered. Indeed, in case of a conformally non-flat background metric ω, the Lie form of $\widehat{\mathcal{R}}_2$ given by the set of equations (15), (16) and (17), is equivalent

to the set of equations made of the two first equations in formula (22) together with the equation (6). Then setting $\widehat{\rho}=\tau=0$ (i.e. a conformally flat metric ω), doesn't change this equivalence, but in that case the equation (6) becomes equivalent to equation (11), and thus the "extended form" of $\widehat{\mathcal{R}}_2$ is the formula (22) and we have one equation more than in the Lie form case. The latter point is rather important to make the difference between the two forms. Obviously the groupoïd $\mathcal{R}_2 \subset \widehat{\mathcal{R}}_2$ of the Poincaré pseudogroup corresponds to the case for which $[\alpha_2]=0$. The symbol M_2 of $\mathcal{R}_2$ vanishes and so is involutive, and $\mathcal{R}_2$ is not formally integrable unless ρ_s is a constant. The present suggested model is associated to a particular split exact short sequence of groupoïds (not of Lie groupoïds because of $\not\mathcal{R}_{_2}$):

$$1 \longrightarrow \mathcal{R}_2 \longrightarrow \widehat{\mathcal{R}}_2 \longrightarrow \not\mathcal{R}_{_2} \longrightarrow 1,$$

and in order to have a relative exact non-linear (even so fractional!) complex associated to $\not\mathcal{R}_{_2}$, the complex associated to $\mathcal{R}_2$ would also have to be exact. This is possible only if $\mathcal{R}_2$ is formally integrable and consequently involutive ($M_2=0$). Setting these conditions it follows that the relation (12) must be satisfied at the sheaf level. Then from relation (12) and (22), one deduces $\not\mathcal{R}_{_2}$ is the set of elements $[\alpha_2]\in \underline{J_2}$ such as $\forall\, X,Y \in T\mathcal{M}$ and $\forall\, c_0 \in \mathbb{R}$:

$$\not\mathcal{R}_{_2}: \qquad \left\{ \begin{array}{rcl} [\mu](X,Y) & = & \frac{1}{2}\Big\{\Big[c_0\left(1-e^{2[\alpha]}\right)-[\beta](_*[\beta])\Big]\,\omega(X,Y) \\ & & +[\beta]\left(\nabla_X Y+\nabla_Y X\right)\Big\} - [\beta](X)[\beta](Y), \end{array}\right. \tag{23}$$

and only with these conditions does $\widehat{\mathcal{R}}_2$ reduce indeed to the system chosen by J.-F. Pommaret. Thus, $[\mu]$ is completly defined from $[\alpha]$ and $[\beta]$ so that the symbol $\not M_{_2}$ of $\not\mathcal{R}_{_2}$ obviously vanishes. Hence, $\not\mathcal{R}_{_2}$ is involutive and one has the equivalence:

$$\not\mathcal{R}_{_2} \simeq \not\mathcal{R}_{_1} = \underline{J_1},$$

deduced from the embedding of $\underline{J_1}$ in $\underline{J_2}$ defined by the system (23). Consequently, one has to work out the complex associated to $\not\mathcal{R}_{_1}$ such that:

$$1 \longrightarrow \mathcal{R}_2 \longrightarrow \widehat{\mathcal{R}}_2 \xrightarrow{\ \phi_0\ } \not\mathcal{R}_{_1} \longrightarrow 1.$$

The sequences above are sequences of groupïds but not of Lie groupoïds. That $\not\mathcal{R}_{_2} \equiv \underline{J_1}$ is a groupoïd can be seen directly from the definition of ϕ_0 as we shall see in the sequel or first by considering locally, above each pair of open subsets $U\times V \subset \mathcal{M}\times\mathcal{M}$ the corresponding associated trivial local groupoïds $U\times G_{U\times V}\ \times\ V \simeq\ \mathcal{R}_{2/U\times V}$ (and with analogous expressions for the other $\mathcal{R}$'s). Then we obtain a corresponding sequence of algebraic groups on a finite projective free module for the G's. Since they are algebraic they are splittable and the sequence is split exact. Therefore we can find a splitting of groups (such as an Iwasawa decomposition for instance), by a good choice of back-connection. Then $\not\mathcal{R}_{_{1/U\times V}}$ can be canonically injected in $\widehat{\mathcal{R}}_{2/U\times V}$ so that it acquires locally an algebraic group structure. Then, by gluing over all pairs of open subsets $U\times V$, we deduce the sequence of groupoïds. Let us add from the definition of $\widehat{\mathcal{R}}_2$ that $\not\mathcal{R}_{_1}$ is a natural bundle associated to $\widehat{\mathcal{R}}_2$. From a physical point of view, it is important to notice

that μ may be considered as an Abraham-Eötvos type tensor, leading to a first physical interpretation (up to a constant for units) of β as the acceleration 4-vector of gravity. On the other hand, completly in agreement with J.-F. Pommaret, α being associated with the dilatations it can be considered as a relative Thomson type temperature (again up to a constant for units):

$$\alpha = \ln(T_0/T), \tag{24}$$

where T_0 is a constant temperature of reference associated with the base space-time $\mathcal{M}$. Then, one can easily define the epimorphism ϕ_0 by the relations $\forall\,[\hat{f}_2] \in \widehat{\mathcal{R}}_2$:

$$\phi_0([\hat{f}_2]) = \begin{cases} [\alpha] & = & \frac{1}{n}\ln|\det([\hat{f}_1])| \\ [\beta] & = & \frac{1}{n}Tr_\omega\left({}^{[\hat{f}_2]}\nabla - \nabla\right). \end{cases} \tag{25}$$

Thus, we get a first diagram:

$$\begin{tikzcd} & 1 \arrow[d] & 1 \arrow[d] \\ 1 \arrow[r] & \Gamma_G \arrow[r, "j_2"] \arrow[d] & \mathcal{R}_2 \arrow[d] \\ 1 \arrow[r] & \Gamma_{\widehat{G}} \arrow[r, "j_2"] \arrow[d] & \widehat{\mathcal{R}}_2 \arrow[d, "\phi_0"] \\ 1 \arrow[r] & \theta \arrow[r, "j_1"] & \underline{J_1} \arrow[d] \\ & & 0 \end{tikzcd} \tag{26}$$

Before presenting the various non-linear Spencer complexes we shall recall briefly certain definitions and results of this theory. The Spencer Theory presentation we give below is rather minimal since we think that it is impossible to describe it perfectly in few pages. It is especially a matter of indicating the notations chosen in the text and we do not claim to make a full and complete description. Moreover, this theory is presented and taken up historically with the "diagonal method" of Grothendieck (Kumpera *et al.* 1972), and the results and definitions we give don't mention it. In this method, two copies of the base space $\mathcal{M}$ are used. The first one $\mathcal{M}_1$ (the horizontal component) is attributed to the set of "points" on which the Taylor coefficients of particular Taylor series are defined, and the second $\mathcal{M}_2$ (the vertical component), to points on which these previous series are evaluated. The independence of this evaluation with respect to the points chosen in $\mathcal{M}_1$ allow to deduce the first Spencer differential operator (linear or non-linear) as an exterior differential operator on the horizontal component. Then the second Spencer differential

operator and a particular set of derivations are deduced from the equivariance of the first Spencer differential operator with respect to a particular groupoïd action. Thus, the Spencer cohomology can be seen mainly as an equivariant cohomology on graded sheafs of the diagonal of $\mathcal{M}_1 \times \mathcal{M}_2$. Actually, we give the results and formulas after the vertical (diagonal isomorphism) projection on the vertical component of various diagonal graded sheafs defined on $\mathcal{M}_1 \times \mathcal{M}_2$ and following in parts a formulation given by J.-F. Pommaret especially when concerning the definition of the differential bracket. The other definitions are merely the vertical "translations" of the definitions given by Kumpera and Spencer applying the so-called "ϵ" isomorphism on the diagonal sheafs.

3.3. The first non-linear Spencer differential operators

First, we recall that $\underline{\Lambda\, T^*\mathcal{M}} \otimes_\theta \underline{J_k(E)}$, where E is a vector bundle, has a natural left $\underline{\Lambda\, T^*\mathcal{M}}$-module structure.

Definition 1 *a) The linear Spencer operator D is the unique differential operator ($\mathbb{R}$-linear sheaf map; $k, s \geq 0$)*

$$D : \underline{\Lambda\, T^*\mathcal{M}} \otimes_\theta \underline{J_{k+1}(E)} \longrightarrow \underline{\Lambda\, T^*\mathcal{M}} \otimes_\theta \underline{J_k(E)},$$

satisfying the three following conditions:

1. $D \circ j_{k+1} = 0$,
2. $\forall \tau_{k+1} \in \underline{\Lambda\, T^*\mathcal{M}} \otimes_\theta \underline{J_{k+1}(E)}$, $\quad D(\omega \wedge \tau_{k+1}) = \mathbf{d}\omega \wedge \tau_k + (-)^{d^0\omega} \omega \wedge D\tau_k$ *where $\omega \in \underline{\Lambda\, T^*\mathcal{M}}$ is any homogeneous differential form and $\mathbf{d}$ being the exterior differential,*
3. *D restricted to $J_{k+1}(E)$ satisfies:*

$$\epsilon_1 \circ D = j_1 \circ \Pi_k^{k+1} - id_{J_{k+1}(E)},$$

where ϵ_1 is the monomorphism defined by the short exact sequence:

$$0 \longrightarrow T^*\mathcal{M} \otimes J_k(E) \xrightarrow{\epsilon_1} J_1(J_k(E)) \xrightarrow{\Pi_0^1} J_k(E) \longrightarrow 0.$$

b) The restriction of $(-D)$ to the symbol $\underline{\Lambda\, T^*\mathcal{M}} \otimes_\theta \underline{S_k T^*\mathcal{M}} \otimes_\theta \underline{E}$ defines the $\underline{\Lambda\, T^*\mathcal{M}}$-linear Spencer map $\underline{\delta}$ such that

$$\underline{\delta}(\omega \wedge \tau_k) = (-)^{d^0\omega} \omega \wedge \underline{\delta}(\tau_k),$$

with ω and τ_k as in a)-3..

■

Definition 2 *We define the r-th Spencer sheaf ($r \geq 1$) of $J_k(E)$, the quotient sheaf*

$$\mathcal{C}_k^r(E) = \underline{\Lambda^r\, T^*\mathcal{M}} \otimes_\theta \underline{J_k(E)} / \zeta_k \circ \underline{\delta}(\underline{\Lambda^{r-1}\, T^*\mathcal{M}} \otimes_\theta \underline{S_{k+1} T^*\mathcal{M}} \otimes_\theta \underline{E}),$$

where ζ_k is the monomorphism defined by the short exact sequence:

$$0 \longrightarrow S_k T^*\mathcal{M} \otimes E \xrightarrow{\zeta_k} J_k(E) \xrightarrow{\Pi_{k-1}^k} J_{k-1}(E) \longrightarrow 0.$$

$\blacksquare$

Let us add that $\mathcal{C}_k^r(E)$ has a module structure on the θ-algebra $\underline{J_k}$ and we set $\mathcal{C}_k^r(E) = 0$ for $r > n$ and $\mathcal{C}_k^0(E) = \underline{J_k(E)}$.

Definition 3 *The operator D can be factorized with a right θ-linear operator D' on $\mathcal{C}_k^r(E)$:*

$$D' : \mathcal{C}_k^r(E) \longrightarrow \mathcal{C}_k^{r+1}(E).$$

$\blacksquare$

Contrary to the operator D, there is no loss of order on k. More precisely, D' is such that the following diagram of split exact sequences is commutative:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\underline{S_{k+1} T^*\mathcal{M}} \otimes_\theta \underline{E} \arrow[r, "(-\delta)"] \arrow[d, "\zeta_{k+1}"] & \underline{T^*\mathcal{M}} \otimes_\theta \underline{S_k T^*\mathcal{M}} \otimes_\theta \underline{E} \arrow[d, "id \otimes \zeta_k"] \\
\underline{J_{k+1}(E)} \arrow[r, "D"] \arrow[d, "\Pi_k^{k+1}"] & \underline{T^*\mathcal{M}} \otimes_\theta \underline{J_k(E)} \arrow[d, "\rho_k^{k+1}"] \\
\underline{J_k(E)} \arrow[r, "D'"] \arrow[d] & \mathcal{C}_k^1(E) \arrow[d] \\
0 & 0
\end{tikzcd}$$

It is to be noted that the sequences being split, then D' is built up from a connexion

$$c_{k+1}^k : \underline{J_k(E)} \longrightarrow \underline{J_{k+1}(E)},$$

such that by definition

$$\Pi_k^{k+1} \circ c_{k+1}^k = id_k.$$

But quotienting, then by definition, D' is independent of the choice of connexion c^k_{k+1}. Hence, whatever is c^k_{k+1}, one has

$$D' = \rho^{k+1}_k \circ D \circ c^k_{k+1}.$$

Finally, for $r \geq 1$, these definitions can be extended to the tangent bundle R_k of $\mathcal{R}_k$ instead of $J_k(E)$, and to its corresponding symbol M_k. But in this case, from the definition of D', 1) $\mathcal{R}_{k+1}$ must be a fibered manifold, 2) to make a choice of connexion c^k_{k+1} we must have the epimorphism $\mathcal{R}_{k+1} \longrightarrow \mathcal{R}_k \longrightarrow 0$, and 3) the system $\mathcal{R}_k$ must be formally transitive, i.e. we must have the epimorphism $\mathcal{R}_k \longrightarrow \mathcal{M} \longrightarrow 0$. Also we shall use the definitions and notations:

$$\mathcal{C}^r_k = \underline{\Lambda^r\, T^*\mathcal{M}} \otimes_\theta R_k / \zeta_k \circ \underline{\delta}(\underline{\Lambda^{r-1}\, T^*\mathcal{M}} \otimes_\theta M_{k+1}).$$

Definition 4 *Let us define $B^r_k(\mathcal{M})$ and $B_k(\mathcal{M})$ such that*

$$B^r_k(\mathcal{M}) = \underline{\Lambda^r\, T^*\mathcal{M}} \otimes_\theta \underline{J_k},$$

and

$$B_k(\mathcal{M}) = \overset{r}{\oplus} B^r_k(\mathcal{M}).$$

■

$B_k(\mathcal{M})$ has a natural structure of graded $\underline{J_k}$-algebra defined by the exterior product of the J_k-valued forms, since we have the equivalence:

$$B_k(\mathcal{M}) \equiv \underline{\Lambda\, T^*\mathcal{M}} \otimes_\theta \underline{J_k}.$$

$B_k(\mathcal{M})$ inherits also a natural structure of left graded $\Lambda\, T^*\mathcal{M}$-module where the external operation on $\overset{r>0}{\oplus} B^r_k(\mathcal{M})$ is the exterior product $\omega \wedge \mu$ of $\omega \in \Lambda\, T^*\mathcal{M}$ and $\mu \in B_k(\mathcal{M})$ with respect to the pairing on $B^0_k(\mathcal{M})$:

$$([f], [g_k]) \in \theta \times \underline{J_k} \longrightarrow j_k([f]).[g_k] \in \underline{J_k}.$$

Let us denote by $Der_a B_k(\mathcal{M})$ the sheaf of germs of "admissible" graded derivations $\mathcal{D}$ of $B_k(\mathcal{M})$, i.e. if $\mathcal{D}$ is of degree p, one has:

$$\mathcal{D}(\underline{J^0_k}) \subset \underline{\Lambda^p\, T^*\mathcal{M}} \otimes_\theta \underline{J^0_k},$$

$$\mathcal{D}(\underline{\Lambda^r\, T^*\mathcal{M}}) \subset \underline{\Lambda^{r+p}\, T^*\mathcal{M}},$$

where $\underline{J^0_k}$ is the kernel of the target map β_k defined on $\underline{J_k}$:

$$\beta_k : \underline{J_k} \longrightarrow \theta.$$

Obviously $Der_a B_k(\mathcal{M})$ is endowed with the bracket of derivations $[\quad , \quad]$, i.e. if $\mathcal{D}_i$ is a derivation of degree p_i, then we have:

$$[\mathcal{D}_1, \mathcal{D}_2] = \mathcal{D}_1 \circ \mathcal{D}_2 - (-)^{p_1 p_2} \mathcal{D}_2 \circ \mathcal{D}_1.$$

Finally, we also have

$$Der_a B_k(\mathcal{M}) = \overset{r}{\oplus} Der_a^r B_k(\mathcal{M}),$$

where $Der_a^r B_k(\mathcal{M})$ is the module of admissible derivations of degree r on $B_k(\mathcal{M})$.

Definition 5 *One defines $\mathbf{D}_1$ (the "twisting" of $\mathbf{d}$), the non-linear differential operator*

$$\mathbf{D}_1 : \Gamma_{k+1}\mathcal{M} \longrightarrow Der_a^1 B_k(\mathcal{M})$$

such that:

$$\mathbf{D}_1[f_{k+1}] = \mathbf{d} - Ad[f_{k+1}] \circ \mathbf{d} \circ Ad[f_{k+1}^{-1}],$$

where $Ad[f_{k+1}]$ is the contravariant action at the sheafs level of $[f_{k+1}]$ on $\underline{\Lambda\, T^*\mathcal{M}} \otimes_\theta \underline{J_k}$ corresponding to the action of the pull-back of $f \in Aut(\mathcal{M})$ on the tensors of $\Lambda\, T^*\mathcal{M} \otimes J_k$. ■

To this action on $\Lambda\, T^*\mathcal{M} \otimes J_k$ corresponds simultaneously an action of the "pull-back-push-forward of $f \in Aut(\mathcal{M})$ " on the tensors of $\Lambda\, T^*\mathcal{M} \otimes J_k(\mathcal{M})$. Also, we deduce and define at the sheaf level, the extension of $Ad[f_{k+1}]$ on $\underline{\Lambda\, T^*\mathcal{M}} \otimes_\theta \underline{J_k(\mathcal{M})}$.

Definition 6 *Let*

$$\mathbf{D'}_1 : \Gamma_k\mathcal{M} \longrightarrow \mathcal{C}_k^1(T\mathcal{M}),$$

be "the first non-linear Spencer operator" such that $\forall [f_k] \in \Gamma_k\mathcal{M}$:

$$\mathbf{D'}_1([f_k]) = \rho_k^{k+1} \circ \left[\mathbf{d} - Ad[f_{k+1}] \circ \mathbf{d} \circ Ad[f_{k+1}^{-1}]\right](id_k), \tag{27}$$

where $[f_{k+1}] = c_{k+1}^k([f_k])$ and $id_k \in \underline{J_k(\mathcal{M})}$ is the prolongation up to order k of $id_\mathcal{M} \equiv id_0$. ■

Sometimes this definition is given using the functor j_1 instead of $\mathbf{d}$. The result is that the derivation is made with respect to the source and not the target and difficulties appear when defining the brackets given further. Then Spencer defined the isomorphism ad of degree zero:

Definition 7 *One defines the isomorphism ad, the operator*

$$ad : \mathcal{C}_{k+1}^\bullet(T\mathcal{M}) \longrightarrow Der_a B_k(\mathcal{M})$$

such that $\forall v \in B_k(\mathcal{M})$ and $\forall u_{k+1} \in \mathcal{C}_{k+1}^r(T\mathcal{M})$:

1. $ad(\mathcal{C}_{k+1}^\bullet(T\mathcal{M})) \overset{def.}{\equiv} Der_\Sigma B_k(\mathcal{M}) = Der_{\Sigma,k}^\bullet \subseteq Der_a B_k(\mathcal{M})$,

2. $\mathcal{L} : \underline{\Lambda T^*\mathcal{M}} \otimes_\theta \underline{J_k(T\mathcal{M})} \longrightarrow Der_a B_k(\mathcal{M})$ *being the Lie derivative of degree zero such that*

$$\mathcal{L}(u_k)v = [i(u_k), \mathbf{d}]v \equiv u_k \barwedge \mathbf{d}v + (-)^r \mathbf{d}(u_k \barwedge v),$$

where $\barwedge$ *is the extended Frölicher-Nijenhuis product and* i *the interior product:*

$$i(u_k)v \equiv u_k \barwedge v,$$

3. $ad(u_{k+1})v = \left[\mathcal{L}(u_k) + (-)^{r+1} D'(u_{k+1})\barwedge\right] v.$

■

Initially, ad is the θ-linear sheafs map corresponding to the $(k+1)$-st order differential operator $\mathcal{L} \circ j_k$ where $\mathcal{L} : J_k(T) \longrightarrow Der_a B_k(\mathcal{M})$ is the Lie derivative of degree zero and of order 1. Then, one obtains the first step of the sophisticated non-linear Spencer complexes associated to the resolutions $\mathcal{C}^\bullet_{k+1}(T\mathcal{M})$ or $Der^\bullet_{\Sigma,k}$:

$$\begin{tikzcd}
1 \arrow[r] & Aut(\mathcal{M}) \arrow[r, "j_{k+1}"] \arrow[d, equal] & \Gamma_{k+1}\mathcal{M} \arrow[r, "\mathbf{D'}_1"] \arrow[d, equal] & \mathcal{C}^1_{k+1}(T\mathcal{M}) \arrow[d, "ad"] \\
1 \arrow[r] & Aut(\mathcal{M}) \arrow[r, "j_{k+1}"] & \Gamma_{k+1}\mathcal{M} \arrow[r, "\mathbf{D}_1"] & Der^1_{\Sigma,k}
\end{tikzcd}$$

where the two horizontal sequences are split exact at $\Gamma_{k+1}\mathcal{M}$. From this, we can determine the induced fractional differential operator $\not{\mathbf{D}}'_1$ such that the following diagram is commutative ($M_3 = \widehat{M}_3 = 0$, $\not{M}_2 = 0$):

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 0 \arrow[d] & \\
1 \arrow[r] & \Gamma_G \arrow[r, "j_2"] \arrow[d] & \mathcal{R}_2 \arrow[r, "\mathbf{D'}_1"] \arrow[d] & \mathcal{C}^1_2 \arrow[d] & = \underline{T^*\mathcal{M}} \otimes_\theta R_2 \\
1 \arrow[r] & \Gamma_{\widehat{G}} \arrow[r, "j_2"] \arrow[d] & \widehat{\mathcal{R}}_2 \arrow[r, "\widehat{\mathbf{D}}'_1"] \arrow[d, "\phi_0"] & \widehat{\mathcal{C}}^1_2 \arrow[d, "\phi_1"] & = \underline{T^*\mathcal{M}} \otimes_\theta \widehat{R}_2 \\
1 \arrow[r] & \theta_{\mathcal{M}} \arrow[r, "j_1"] & \underline{J_1} \arrow[r, "\not{\mathbf{D}}'_1"] \arrow[d] & \not{\mathcal{C}}^1_1 \arrow[d] & = B^1_1(\mathcal{M}) \\
 & & 1 & 0 &
\end{tikzcd} \tag{28}$$

Obviously, one will determine also ϕ_1 and in the sequel, one will call $B^1_1(\mathcal{M})$ the sheaf of electromagnetic and gravitational gauge potentials.

4. The potentials of interaction and the metric

4.1. The electromagnetic and gravitational potentials

In order to lighten the presentation of the results, first let us consider the following notations:

1) one will merely write f_k and $j_k(f)$ instead of respectively $[f_k]$ and $j_k([f])$,
2) one will denote $T^q f$ the restriction in $\underline{S_q T^*\mathcal{M}} \otimes_\theta \underline{T\mathcal{M}}$ of $[f_p] \in \underline{J_p(\mathcal{M})}$ $(p \geq q \geq 1)$,
3) one will denote $T(T^r f)$ the restriction in $\underline{T^*\mathcal{M}} \otimes_\theta \underline{S_r T^*\mathcal{M}} \otimes_\theta \underline{T\mathcal{M}}$ of $j_1([f_s]) \in \underline{J_1(J_s(\mathcal{M}))}$ $(s \geq r \geq 0)$,
4) one sets analogous conventions concerning the presence of ordinary parenthesis in the notations for the differential of germs of the tangent maps $dT^q f$ and $d(T^q f)$ (let us note in these notations that d is not the exterior differential that one denotes by $\mathbf{d}$, but stands for differential maps).

Then, one has the following set of results $\forall\, \hat{f}_2 \in \widehat{\mathcal{R}}_2$ and $\forall\, X \in \mathcal{C}^1(T\mathcal{M})$:

$$
\begin{aligned}
Tr^1({}^{j_1(\hat{f}_1)}\nabla_X) &= \frac{1}{2}\sum_{i,j=1}^{n} \tilde{\omega}^{i,j} \Big\{ << d(\omega)\circ\hat{f} | T(\hat{f}).X > | T\hat{f}.e_i \otimes T\hat{f}.e_j > \\
&\quad + 2 < \omega\circ\hat{f} | < d(T\hat{f}) | X > .e_i \otimes T\hat{f}.e_j > \Big\} && (29) \\
&= Tr^1(\nabla_X) + n < \mathbf{d}(\alpha) | X >, && (30)
\end{aligned}
$$

and

$$
\begin{aligned}
Tr^1({}^{\hat{f}_2}\nabla_X) &= \frac{1}{2}\sum_{i,j=1}^{n} \tilde{\omega}^{i,j} \Big\{ << d(\omega)\circ\hat{f} | T\hat{f}.X > | T\hat{f}.e_i \otimes T\hat{f}.e_j > \\
&\quad + 2 < \omega\circ\hat{f} | < dT\hat{f} | X > .e_i \otimes T\hat{f}.e_j > \Big\} && (31) \\
&= Tr^1(\nabla_X) + n < \beta | X > . && (32)
\end{aligned}
$$

Now, let $\widehat{\chi}^{(2)}$ be an element of $\underline{J_1(J_2(T\mathcal{M}))}$, and its components $\widehat{\chi}_q$ $(q = 0, 1, 2)$, i.e. the restrictions of $\widehat{\chi}^{(2)}$ to $\underline{T^*\mathcal{M}} \otimes_\theta \underline{S_q T^*\mathcal{M}} \otimes_\theta \underline{T\mathcal{M}}$ such that $\forall \hat{f}_3 \in \widehat{\mathcal{R}}_3$:

$$\widehat{\chi}^{(2)} = \hat{f}_3^{-1} \circ j_1(\hat{f}_2) - id_3, \qquad (33)$$

where by abuse of notations id_3 is the image of $id_3 \in J_3(T\mathcal{M})$ by the canonical injection $J_3(T\mathcal{M}) \longrightarrow J_1(J_2(T\mathcal{M}))$. In particularly, one has the relation (Pommaret 1994):

$$
\begin{aligned}
\widehat{\chi}_0 &= \hat{f}_1^{-1} \circ j_1(\hat{f}) - id_1 \\
&\equiv \widehat{A} - id_1.
\end{aligned}
$$

It follows that $\widehat{\mathbf{D}}'_1$ satisfies:

$$\epsilon_1 \circ \widehat{\mathbf{D}}'_1(\hat{f}_2) \equiv \widehat{\tau}^{(2)} = \widehat{\chi}^{(2)} \circ (\widehat{B} \otimes id_2), \tag{34}$$

with $\widehat{B} = \widehat{A}^{-1}$ and $\hat{f}_3 = c_3^2(\hat{f}_2) \in \widehat{\mathcal{R}}_3$. In particular, $\widehat{\tau}_0$ and $\widehat{\tau}_1$ satisfy the relations:

$$\widehat{\tau}_0 = id_{T\mathcal{M}} - T\hat{f} \circ T(\hat{f})^{-1} \equiv id_{T\mathcal{M}} - \widehat{B} = \widehat{\chi}_0 \,.\, \widehat{B} \in T\mathcal{M} \otimes T^*\mathcal{M},$$

and $\forall\, X, Y \in T\mathcal{M}$ (Pommaret 1994):

$$< dT\hat{f}|X > \circ\, \widehat{\tau}_0 \,.\, Y + T\hat{f}\circ < \widehat{\tau}_1|Y > .\, X =< d(T\hat{f}) - dT\hat{f}|\widehat{B}.Y > .\, X,$$

where $< dT\hat{f}|X >$, $< \widehat{\tau}_1|Y > \equiv < \widehat{\chi}_1|\widehat{B}.Y >$ and $< d(T\hat{f}) - dT\hat{f}|\widehat{B}.Y >$, are considered as elements of $T\mathcal{M} \otimes T^*\mathcal{M}$. Then, if one substitutes

$$T\hat{f}^{-1} \circ T(\hat{f}).X = (\widehat{\chi}_0 + id).X$$

for X in the relations (31)-(32), and also considering the Schwarz equalities:

$$< dT\hat{f}|X > .\, Y =< dT\hat{f}|Y > .\, X,$$

one obtains by subtracting the result from the relations (29)-(30) $\forall\, X \in T\mathcal{M}$:

$$\frac{1}{n} Tr^1 \left[< \widehat{\chi}_1|X > + \nabla_{\widehat{\chi}_0 . X}\right] =< \mathbf{d}(\alpha) - \beta|X > - < \beta|\widehat{\chi}_0.X > . \tag{35}$$

From this latter relation, one can define the electromagnetic potential vector $\widehat{\mathcal{A}} \in \underline{T^*\mathcal{M}}$ by $\forall\, X \in \underline{T\mathcal{M}}$:

$$< \widehat{\mathcal{A}}|X > = \frac{1}{n} Tr^1 \left[< \widehat{\tau}_1|X > + \nabla_{\widehat{\tau}_0 . X}\right] =< \mathbf{d}(\alpha)|\widehat{B}.X > - < \beta|X > . \tag{36}$$

In an orthonormal system of coordinates, the latter definition becomes ($i, j, k = 1, ..., n$):

$$\widehat{\mathcal{A}}_i = \frac{1}{n} \left(\widehat{\tau}^k_{k,i} + \widehat{\tau}^k_{,i} \gamma^j_{j\,k}\right) = \widehat{B}^k_i \partial_k \alpha - \beta_k,$$

where γ is the Riemann-Christoffel 1-form associated to ω and thus satisfying:

$$Tr^1(\nabla_X) = Tr^1(\gamma(X)).$$

Prolonging the relation (35) (one does not prolong the relation (36) because $\widehat{\tau}_2$ is not the first prolongation of $\widehat{\tau}_1$, contrary to $\widehat{\chi}_2$), one deduces and defines the mixed tensor potential of gravitation and electromagnetic $\widehat{\mathcal{B}} \in \overset{2}{\otimes} \underline{T^*\mathcal{M}}$ such as $\forall\, X, Y \in T^*\mathcal{M}$:

$$\begin{aligned} < \widehat{\mathcal{B}}|Y \otimes X > &= \frac{1}{n} Tr^1 \left[i_Y < \widehat{\tau}_2|X > + < d(\gamma)|Y \otimes \widehat{\tau}_0.X > \right. \\ &\quad \left. + \gamma(< \widehat{\tau}_1|X > .\, Y) \right] \\ &= < \mathbf{d}(\beta)|\widehat{B}.X \otimes Y > - < \beta| < \widehat{\tau}_1|X > .\, Y > \\ &\quad - < \mu|X \otimes Y > \end{aligned} \tag{37}$$

where i_Y is the interior product by Y and $\widehat{\tau}_2$ satisfies $\forall\,\hat{f}_3 \in \widehat{\mathcal{R}}_3$ and $\forall\, X, Y, Z \in T\mathcal{M}$:

$$<< d(dT\hat{f}) - d^2T\hat{f}|\widehat{B}.X > |Y \otimes Z >=$$

$$\begin{array}{l} < d^2T\hat{f}|Y \otimes Z > \circ\, \widehat{\tau}_0 \,.\, X + < dT\hat{f}|Y > \circ < \widehat{\tau}_1|X > .\, Z \\ + < dT\hat{f}|Z > \circ < \widehat{\tau}_1|X > .\, Y + T\hat{f}\circ << \widehat{\tau}_2|X > |Y \otimes Z > . \end{array}$$

Again, in an orthonormal system of coordinates $(i, j, k, h = 1, ..., n)$:

$$\widehat{\mathcal{B}}_{j,i} = \frac{1}{n}\left(\widehat{\tau}^{k}_{k\,j,i} + \widehat{\tau}^{k}_{j\,,i}\gamma^{h}_{h\,k} + \widehat{\tau}^{k}_{,i}(\partial_k\gamma^{h}_{h\,j})\right) = \widehat{B}^{k}_{i}\partial_k\beta_j - \mu_{i\,j} - \widehat{\tau}^{k}_{j\,,i}\beta_k.$$

This definition for $\widehat{\mathcal{B}}$ hasn't been determined by J.-F. Pommaret (1994), he also gave a different definition for $\widehat{\mathcal{A}}$ rather associated to the relation (35). In conclusion to this chapter, one notices that $\not{\mathbf{D}}'_{\mathbf{1}}$ (remaining to explicit) appears to be a Fröbenius-type operator and depends on $\widehat{\tau}^{(1)}$ itself depending on $\widehat{\mathcal{A}}$ and $\widehat{\mathcal{B}}$ as we shall see further. Lastly, ϕ_1 is quite defined by the relations (36) and (37), and if $n = 4$, one has 20 scalar gauge potentials. On the other hand, one can see as well that the definitions of $\widehat{\mathcal{A}}$ and $\widehat{\mathcal{B}}$ can be deduced from the conformal Killing equations on $\underline{J_1(T\mathcal{M})}$, namely $\forall\,\xi_{(1)} \equiv (\xi_0, \xi_1) \in \underline{J_1(T\mathcal{M})}$ and $\forall\,\eta \in \theta$ then one has:

$$K_0(\xi^{(1)}) \stackrel{def.}{\equiv} \frac{1}{n}Tr^1(\xi_1 + \gamma(\xi_0)) = \eta,$$

where K_0 is the conformal Killing operator. If K_1 is its first prolongation then setting $K^{(1)} \equiv (K_0, K_1)$ one obtains obviously from the equation above $\forall\, X \in \underline{T\mathcal{M}}$ and $\forall\,\widehat{\tau}^{(2)} \in \widehat{\mathcal{C}}^1_2(T\mathcal{M})$:

$$\phi_1(\widehat{\tau}^{(2)})(X) = K^{(1)}(< \widehat{\tau}^{(2)}|X >).$$

4.2. The morphisms ϕ_1, $\not{\mathbf{D}}'_{\mathbf{1}}$ and the metric of the "gauge space-time"

From the preceding chapter, obviously one easily notices by definition that $\mathcal{C}^1_2$ is the kernel of ϕ_1. On the contrary $\not{\mathbf{D}}'_{\mathbf{1}}$, depending on $\widehat{\tau}^{(1)}$, is rather more tricky to determine. The sequences being split and ϕ_1 being θ-linear, one can deduce the important relation that $\widehat{\tau}^{(2)}$ can be rewritten in the following form:

$$\widehat{\tau}^{(2)} = \tau^{(2)} + < \widehat{\mathcal{A}}|\Omega^{(2)} > + < \widehat{\mathcal{B}}|\mathcal{K}^{(2)} >, \tag{38}$$

where $\tau^{(2)} \in \mathcal{C}^1_2$ and $\Omega^{(2)}$ and $\mathcal{K}^{(2)}$ are elements of $\underline{\Lambda\,T^*\mathcal{M}}\otimes_\theta(\widehat{\mathcal{C}}^1_2\big/\mathcal{C}^1_2)$, one calls the linear susceptibilities of the vacuum or the space-time associated respectively to $\widehat{\mathcal{A}}$ and $\widehat{\mathcal{B}}$. Then, as a definition of $\not{\mathbf{D}}'_{\mathbf{1}}$, one obtains the following kind of relations:

$$\begin{array}{rcccl} < a_1|\widehat{\mathcal{A}} > + < b_1|\widehat{\mathcal{B}} > & = & D'\alpha & \equiv & \mathbf{d}\alpha - \beta, \\ < a_2|\widehat{\mathcal{A}} > + < b_2|\widehat{\mathcal{B}} > & = & D'\beta & \equiv & \mathbf{d}\beta - \mu, \end{array}$$

where a_1 is affine with respect to $\mathbf{d}\alpha$, b_1 linear with respect to $\mathbf{d}\alpha$, a_2 linear with respect to $\mathbf{d}\beta$ and β, and b_2 affine with respect to $\mathbf{d}\beta$ and β. Thus, $\not{\mathbf{D}}'_{\mathbf{1}}$ is fractional with respect to $\mathbf{d}\alpha$, $\mathbf{d}\beta$, α and β and finally only depends on the susceptibilities of $\widehat{\tau}^{(2)}$. Then, the exactness condition at $\underline{J_1}$:

$$\not{\mathbf{D}}'_{\mathbf{1}} \circ j_1 = 0,$$

the relations (36)-(37) and the commutativity of the diagram (28) involve the necessary condition which must be satified by $\tau^{(2)}$:

$$\tau_0 = 0.$$

There remains the metric ν of the "gauge (or observable or measurable) space-time" defined in considering $\widehat{B}$ as a field of tetrads $\forall\, X, Y \in T\mathcal{M}$:

$$< \nu | X \otimes Y > = < \omega | \widehat{B}.X \otimes \widehat{B}.Y > .$$

Thus, one has the general relation between ν and ω:

$$\nu = \omega + \text{linear and quadratic terms in } \widehat{\mathcal{A}} \text{ and } \widehat{\mathcal{B}}.$$

Then from this metric ν, one can deduce the Riemann and Weyl curvature tensors of the gauge space-time. One has a non-metrical theory for the gravitation in the gauge space-time, since clearly ν doesn't appear as a gravitational potential. The space-time terminology we use is quite natural in the sense that one has simultaneously two types of space-time. The first one, which we call the "underlying" or "substrat" space-time, is endowed with the metric ω and is of constant scalar curvature. The other one, called the "gauge (observable or measurable) space-time", endowed with the metric ν, is defined for any scalar curvature and by the gauge potentials $\widehat{\mathcal{A}}$ and $\widehat{\mathcal{B}}$. It can be considered as the underlying space-time deformed by the gauge potentials and the Weyl curvature does not necessarily vanish, contrary to Pommaret's assertions (Pommaret 1994, see page 456 and Pommaret 1989). Moreover, from a continuum mechanics of deformable bodies point of view, the metric ν can be interpreted as the tensor of deformation of the underlying space-time (Katanaev *et al.* 1992, Kleinert 1989).

5. The fields of interaction

5.1. The second non-linear Spencer operator

Before giving an explicit expression for this operator in the complex of $\underline{J_1}$, one will briefly recall its definition, but before that one needs a few other definitions (Pommaret 1989, Kumpera *et al.* 1972).

Definition 8 *Let the "algebraic bracket" $\{\ \ ,\ \ \}$ on $J_{k+1}(T\mathcal{M})$, be the $\mathbb{R}$-bilinear map ($\forall k \geq 0$)*

$$\{\ \ ,\ \ \} : J_{k+1}(T\mathcal{M}) \times_{\mathcal{M}} J_{k+1}(T\mathcal{M}) \longrightarrow J_k(T\mathcal{M})$$

such as $\forall \xi_{k+1}, \eta_{k+1} \in J_{k+1}(T\mathcal{M})$ *one has:*

$$\{\xi_{k+1}, \eta_{k+1}\} = \{\xi_1, \eta_1\}_k,$$

where $\{\xi_1, \eta_1\} \in T\mathcal{M}$ *is the usual Lie bracket defined on* $J_1(T\mathcal{M})$ *and* $\{\xi_1, \eta_1\}_k$ *its lift in* $J_k(T\mathcal{M})$.

■

Definition 9 *One calls "differential Lie bracket" on* $\underline{J_k(T\mathcal{M})}$, *the bracket* $[\![\ ,\ ]\!]$ *such that:*

$$[\![\ ,\ ]\!] : \underline{J_k(T\mathcal{M})} \times_{\mathcal{M}} \underline{J_k(T\mathcal{M})} \longrightarrow \underline{J_k(T\mathcal{M})},$$

and $\forall \xi_k, \eta_k \in \underline{J_k(T\mathcal{M})}$ *then*

$$[\![\xi_k, \eta_k]\!] = \{\xi_1, \eta_1\} + i_{\xi_0} D\eta_{k+1} - i_{\eta_0} D\xi_{k+1},$$

where i *is the usual interior product and* ξ_{k+1} *and* η_{k+1} *are any lifts of* ξ_k *and* η_k *in* $\underline{J_{k+1}(T\mathcal{M})}$.

■

Definition 10 *For any decomposable elements*

$$\alpha = u \otimes \xi_k \in \underline{\Lambda^r T^*\mathcal{M}} \oplus \underline{J_k(T\mathcal{M})},$$

$$\beta = v \otimes \eta_k \in \underline{\Lambda^s T^*\mathcal{M}} \oplus \underline{J_k(T\mathcal{M})},$$

and defining on $\underline{\Lambda T^*\mathcal{M}} \otimes_\theta \underline{J_k(T\mathcal{M})}$ *the interior product* i *by the relation:* $\forall w \in \underline{\Lambda T^*\mathcal{M}}$,

$$i_\alpha w = u \wedge i_{\xi_k} w,$$

then with $\mathbf{d}$ *being the exterior derivative and*

1) $\mathcal{L}$ *the Lie derivative on* $\underline{\Lambda T^*\mathcal{M}} \otimes_\theta \underline{J_k(T\mathcal{M})}$ *such that:*

$$\mathcal{L}(\alpha) = i_\alpha \circ \mathbf{d} + (-)^r \mathbf{d} \circ i_\alpha,$$

2) $ad(\alpha) = \mathcal{L}(\alpha) + (-)^{r+1} i_{D\alpha}$,

one defines the "twisted" bracket on $\underline{\Lambda T^*\mathcal{M}} \otimes_\theta \underline{J_k(T\mathcal{M})}$ *(θ-bilinear),* $[\![\ ,\ ]\!]$ *by the relation:*

$$[\![\alpha, \beta]\!] = [ad(\alpha)v] \otimes \eta_k - (-)^{rs}[ad(\beta)u] \otimes \xi_k + (u \wedge v) \otimes [\![\xi_k, \eta_k]\!] \in \underline{\Lambda^{r+s} T^*\mathcal{M}} \otimes_\theta \underline{J_k(T\mathcal{M})}.$$

■

This bracket defines a graded Lie algebra structure on $\underline{\Lambda T^*\mathcal{M}} \otimes_\theta \underline{J_k(T\mathcal{M})}$.

Definition 11 *One calls 2^{nd} non-linear Spencer operator of the resolution $Der^{\bullet}_{\Sigma,k}$, the differential operator $\mathbf{D}_2$ such that:*

$$\mathbf{D}_2 : Der^1_{\Sigma,k} \longrightarrow Der^2_{\Sigma,k},$$

and $\forall u \in Der^1_{\Sigma,k}$

$$\mathbf{D}_2 u = [\mathbf{d}, u] - \frac{1}{2}[u, u],$$

■

Definition 12 *To this operator $\mathbf{D}_2$ corresponds the 2^{nd} non-linear Spencer operator $\mathbf{D}'_2$ of the resolution $\mathcal{C}^{\bullet}_k(T\mathcal{M})$ such that:*

$$\mathbf{D}'_2 : \mathcal{C}^1_k(T\mathcal{M}) \longrightarrow \mathcal{C}^2_k(T\mathcal{M}),$$

and $\forall \tau \in \mathcal{C}^1_k(T\mathcal{M})$

$$\mathbf{D}'_2\tau = D'\tau - \frac{1}{2}[\![\tau, \tau]\!]',$$

where $[\![\ \ ,\ \ ]\!]'$ is the quotient twisted bracket.

■

Definition 13 *With the latter definitions, the "sophisticated non-linear Spencer complex of $\Gamma_{k+1}(\mathcal{M})$", is the truncated split exact differential sequence in the first row of the following commutative diagram of split exact sequences:*

$$\begin{tikzcd}
1 \arrow[r] & Aut(\mathcal{M}) \arrow[r, "j_{k+1}"] \arrow[d, equal] & \Gamma_{k+1}(\mathcal{M}) \arrow[r, "\mathbf{D}'_1"] \arrow[d, equal] & \mathcal{C}^1_{k+1}(T\mathcal{M}) \arrow[r, "\mathbf{D}'_2"] \arrow[d, "ad"] & \mathcal{C}^2_{k+1}(T\mathcal{M}) \arrow[d, "ad"] \\
1 \arrow[r] & Aut(\mathcal{M}) \arrow[r, "j_{k+1}"] & \Gamma_{k+1}(\mathcal{M}) \arrow[r, "\mathbf{D}_1"] & Der^1_{\Sigma,k} \arrow[r, "\mathbf{D}_2"] & Der^2_{\Sigma,k}
\end{tikzcd}$$

where ad is the isomorphism of graded Lie algebras given in the definition (10), i.e. $\forall \tau, \chi \ \in \ \mathcal{C}^{\bullet}_k(T\mathcal{M})$:

$$ad([\![\tau, \chi]\!]') = [ad(\tau), ad(\chi)].$$

■

These sequences can be restricted to $\widehat{\mathcal{R}}_{k+1}$ if $\widehat{\mathcal{R}}_{k+1}$ satisfies the same properties as those given in the sequel of definition (3) concerning D', and moreover if it is 2-acyclic.

5.2. The gravitational and electromagnetic fields

On the one hand, one has the following commutative diagram:

$$\begin{tikzcd}
\underline{J_1} \arrow[r, "\not{\mathbf{D}}'_{\mathbf{1}}"] & \underline{T^*\mathcal{M}}\otimes_\theta \underline{J_1} \\
\widehat{\mathcal{R}}_2 \arrow[u, "\phi_0"] \arrow[r, "\widehat{\mathbf{D}}'_1"] \arrow[d, equal, "id"] & \widehat{\mathcal{C}}^1_2 \arrow[u, "\phi_1"] \arrow[d, "ad"] \\
\widehat{\mathcal{R}}_2 \arrow[r, "\widehat{\mathbf{D}}_1"] & Der^1_{\Sigma,a}(\widehat{B}_1)
\end{tikzcd}$$

where $\widehat{B}_1 = \underline{\Lambda\, T^*\mathcal{M}}\otimes_\theta \underline{J_1}$, and on the other hand, one obtains the following relation deduced from the relations (36) and (37): $\forall\, \alpha_1 \in \underline{J_1} \equiv \widehat{B}^0_1$,

$$\not{\mathbf{D}}'_{\mathbf{1}}(\alpha_1) = D'(\alpha_1) - \widehat{\mathbf{D}}_1(\hat{f}_2)(\alpha_1).$$

Also let us define $\widehat{\mathfrak{D}}_1 \equiv \widehat{\mathbf{D}}_1(\hat{f}_2) \in Der^1_{\Sigma,a}(\widehat{B}_1)$ and $\mathfrak{D}$ such that:

$$\mathfrak{D} = D' - \widehat{\mathfrak{D}}_1 \in Der^1_{\Sigma,a}(\widehat{B}_1).$$

It follows one can rewrite:

$$\mathfrak{D}\circ\mathfrak{D} = -\widehat{\mathbf{D}}_2(\widehat{\mathfrak{D}}_1),$$

where $\widehat{\mathbf{D}}_2$ is the 2^{nd} non-linear Spencer operator of the sophisticated Spencer complex of $\widehat{\mathcal{R}}_2$:

$$1 \longrightarrow Aut(\mathcal{M}) \longrightarrow \widehat{\mathcal{R}}_2 \xrightarrow{\widehat{\mathbf{D}}_1} Der^1_{\Sigma,a}(\widehat{B}_1) \xrightarrow{\widehat{\mathbf{D}}_2} Der^2_{\Sigma,a}(\widehat{B}_1).$$

Then, from the definition of $Der^\bullet_{\Sigma,k}$, there exists $\widehat{\tau}^{(2)} \in \widehat{\mathcal{C}}^1_2$ such that $\widehat{\mathfrak{D}}_1 = ad(\widehat{\tau}^{(2)})$ and therefore one can write also:

$$\mathfrak{D}\circ\mathfrak{D} = -ad(\widehat{\mathbf{D}}'_2(\widehat{\tau}^{(2)})).$$

Hence $\forall\, \alpha_1 \in \underline{J_1}$ one deduces the relation:

$$\mathfrak{D}\circ\not{\mathbf{D}}'_{\mathbf{1}}(\alpha_1) = -ad(\widehat{\mathbf{D}}'_2(\widehat{\tau}^{(2)}))(\alpha_1),$$

and also $\forall \widehat{\tau}^{(2)} \in \widehat{\mathcal{C}}^1_2$:

$$\begin{aligned}
\mathfrak{D}\circ\phi_1(\widehat{\tau}^{(2)}) &= D'\circ\phi_1(\widehat{\tau}^{(2)}) - \widehat{\mathfrak{D}}_1\circ\phi_1(\widehat{\tau}^{(2)}) \\
&= D'\circ\phi_1(\widehat{\tau}^{(2)}) - ad(\widehat{\tau}^{(2)})\circ\phi_1(\widehat{\tau}^{(2)}) \\
&= D'\circ\tfrac{1}{n}Tr^1(\widehat{\tau}^{(2)}) - ad(\widehat{\tau}^{(2)})\circ\phi_1(\widehat{\tau}^{(2)}) \\
&= \tfrac{1}{n}Tr^1\circ D'(\widehat{\tau}^{(2)}) - ad(\widehat{\tau}^{(2)})\circ\phi_1(\widehat{\tau}^{(2)}) \\
&\overset{def.\,\phi_2}{\equiv} \phi_2\circ D'(\widehat{\tau}^{(2)}) - ad(\widehat{\tau}^{(2)})\circ\phi_1(\widehat{\tau}^{(2)}),
\end{aligned}$$

which can be rewritten:

$$\mathfrak{D} \circ \phi_1(\widehat{\tau}^{(2)}) = \phi_2(\widehat{\mathbf{D}}'_2(\widehat{\tau}^{(2)})) + \frac{1}{2}\phi_2([\![\widehat{\tau}^{(2)}, \widehat{\tau}^{(2)}]\!]) - ad(\widehat{\tau}^{(2)}) \circ \phi_1(\widehat{\tau}^{(2)}),$$

where ϕ_2 is the θ-linear morphism:

$$\phi_2 : \widehat{\mathcal{C}}^2_2 \longrightarrow Der^2_{\Sigma,1} \equiv Der^2_{\Sigma,a}(\widehat{B}_1),$$

satisfying the relations: $\forall\, \widehat{\sigma}^{(2)} \in \widehat{\mathcal{C}}^2_2$,

$$\begin{array}{c} \phi_2(\widehat{\sigma}^{(2)}) \equiv \frac{1}{n} Tr^1(\widehat{\sigma}^{(2)}) \\ \text{and} \quad \phi_2 \circ D' = D' \circ \phi_1. \end{array}$$

Then , defining $\not{\mathbf{D}}'_{\mathbf{2}} \overset{def.}{\equiv} \mathfrak{D} \Big/ _{\underline{T^*\mathcal{M}}\otimes_\theta \underline{J_1}}$, we deduce the theorem:

Theorem *The following diagram of differential sequences is commutative, i.e.* $\not{\mathbf{D}}'_{\mathbf{2}} \circ \phi_1 = \phi_2 \circ \widehat{\mathbf{D}}'_2$*:*

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
1 \arrow[r] & \Gamma_G \arrow[r, "j_2"] \arrow[d] & \mathcal{R}_2 \arrow[r, "\mathbf{D}'_1"] \arrow[d] & \mathcal{C}^1_2 \arrow[r, "\mathbf{D}'_2"] \arrow[d] & \mathcal{C}^2_2 \arrow[d] \\
1 \arrow[r] & \Gamma_{\widehat{G}} \arrow[r, "j_2"] \arrow[d] & \widehat{\mathcal{R}}_2 \arrow[r, "\widehat{\mathbf{D}}'_1"] \arrow[d, "\phi_0"] & \widehat{\mathcal{C}}^1_2 \arrow[r, "\widehat{\mathbf{D}}'_2"] \arrow[d, "\phi_1"] & \widehat{\mathcal{C}}^2_2 \arrow[d, "\phi_2"] \\
1 \arrow[r] & \theta \arrow[r, "j_1"] & \underline{J_1} \arrow[r, "\not{\mathbf{D}}'_{\mathbf{1}}"] \arrow[d] & \underline{T^*\mathcal{M}}\otimes_\theta \underline{J_1} \arrow[r, "\not{\mathbf{D}}'_{\mathbf{2}}"] \arrow[d] & \underline{\Lambda^2\, T^*\mathcal{M}}\otimes_\theta \underline{J_1} \arrow[d] \\
 & & 1 & 0 & 0
\end{tikzcd} \tag{39}$$

if the condition below is satisfied: $\forall\, \widehat{\tau} \in \widehat{\mathcal{C}}^1_2$,

$$\phi_2([\![\widehat{\tau}, \widehat{\tau}]\!]) = 2\, ad(\widehat{\tau}) \circ \phi_1(\widehat{\tau}). \tag{40}$$

$\blacksquare$

Finally, one notices from a diagram chasing in the diagram (39) that the sequence is exact at $\underline{T^*\mathcal{M}}\otimes_\theta \underline{J_1}$ if and only if one restricts to $Im\, \not{\mathbf{D}}'_{\mathbf{1}}$, i.e. one has only the short split exact sequence:

$$1 \longrightarrow \theta \xrightarrow{\ j_1\ } \underline{J_1} \xrightarrow{\not{\mathbf{D}}'_{\mathbf{1}}} \not{\mathbf{D}}'_{\mathbf{1}}(\underline{J_1}) \xrightarrow{\not{\mathbf{D}}'_{\mathbf{2}}} 0. \tag{41}$$

This might be (?) an illustration of the fondamental theorem of Spencer on the deformations of Lie structures (Spencer 1962, 1965, Goldschmidt 1976a, 1976b, 1978a, 1978b, 1981) since $\underline{J_1}$ is endowed with a Lie group structure.

Let us denote $\widehat{\mathcal{C}}$ and $\pi^{(2)}$ by:

$$\begin{aligned}
\widehat{\mathcal{C}} &\equiv (\widehat{\mathcal{A}},\widehat{\mathcal{B}}) \in \widehat{B}_1^1 = \underline{T^*\mathcal{M}}\otimes_\theta \underline{J_1} \\
\pi^{(2)} &\equiv (\Omega^{(2)},\mathcal{K}^{(2)}) \in \widehat{B}_1^{1\,*}\otimes_\theta(\widehat{\mathcal{C}}_2^1/\mathcal{C}_2^1) \\
\text{and} \quad & <\widehat{\mathcal{C}}|\pi^{(2)}> \stackrel{def.}{\equiv} <\widehat{\mathcal{A}}|\Omega^{(2)}> + <\widehat{\mathcal{B}}|\mathcal{K}^{(2)}> = \widehat{\tau} - \tau^{(2)}
\end{aligned}$$

with, as a consequence of the relations (36) and (37):

$$\phi_1(\pi^{(2)}) = 1d_{\widehat{B}_1^1} \in \widehat{B}_1^{1\,*}\otimes_\theta \widehat{B}_1^1 \quad \text{and} \quad \phi_1(\tau^{(2)}) = 0.$$

Then, from (40), one obtains the following equivalent relations:

$$\phi_2([\![\tau^{(2)},\tau^{(2)}]\!]) = 0 \tag{42}$$
$$\phi_2([\![\tau^{(2)},\pi^{(2)})]\!]) = ad(\tau^{(2)}) \tag{43}$$
$$\phi_2([\![\pi^{(2)},\pi^{(2)}]\!]) = 2\,ad(\pi^{(2)}), \tag{44}$$

which are the defining constraints on the $\tau^{(2)}$ and the susceptibilities $\pi^{(2)}$.

In an orthonormal system of local coordinates, one can write $\mathbb{D}'_{\mathbf{2}}(\widehat{\mathcal{C}}) \equiv (\widehat{\mathcal{G}},\widehat{\mathcal{H}}) = \phi_2(\widehat{\sigma}^{(2)}) \in \underline{\Lambda^2\,T^*\mathcal{M}}\otimes_\theta \underline{J_1} = \widehat{B}_1^2$ in the form $(h,i,j,k,l,r,s = 1,...,n)$:

$$\begin{aligned}
\widehat{\mathcal{G}}_{[i\,j]} &= \tfrac{1}{n}(\widehat{\sigma}^k_{k,[i\,j]} + \widehat{\sigma}^k_{,[i\,j]}\gamma^h_{h\,k}) \\
&\equiv \widehat{B}^k_{[i}(\partial_{|\,k\,|}\widehat{\mathcal{A}}_{j\,]}) - \widehat{\mathcal{F}}_{[i\,j]} - \widehat{\tau}^k_{[j,i]}\widehat{\mathcal{A}}_k \\
\widehat{\mathcal{H}}_{j,[k\,i]} &= \tfrac{1}{n}\left(\widehat{\sigma}^h_{h\,j,[k\,i]} + \widehat{\sigma}^l_{j,[k\,i]}\gamma^h_{h\,l} + \widehat{\sigma}^k_{,[k\,i]}(\partial_j\gamma^h_{h\,k})\right) \\
&\equiv \widehat{B}^h_{[k}(\partial_{|\,h}\widehat{\mathcal{B}}_{j,|\,i\,]}) - \widehat{\mathcal{E}}_{j\,[k,i]} - \widehat{\tau}^r_{[i,k]}\widehat{\mathcal{B}}_{j,r} - \widehat{\tau}^r_{j,[\,k}\widehat{\mathcal{B}}_{|\,r,|\,i\,]} - \widehat{\tau}^r_{j\,[i,k]}\widehat{\mathcal{A}}_r
\end{aligned}$$

where

1. $\widehat{\mathcal{F}} \in \Lambda^2\,T^*\mathcal{M}$ is the skew-symmetric part of $\widehat{\mathcal{B}}$ one calls the electromagnetic field tensor (or the Faraday tensor),

2. $(\widehat{\mathcal{E}},\widehat{\mathcal{B}},\widehat{\mathcal{A}}) \in \underline{T^*\mathcal{M}}\otimes_\theta \mathcal{R}_2$, i.e. $\forall\, X,Y,Z \in T\mathcal{M}$, $\forall\, c_0 \in \mathbb{R}$,

$$<\widehat{\mathcal{E}}(X,Y)|Z> = \frac{1}{2} <\widehat{\mathcal{B}}(\nabla_X Y + \nabla_Y X)|Z> - c_0\,\omega(X,Y) <\widehat{\mathcal{A}}|Z>,$$

3. $\widehat{\tau}^{(2)}$ satisfies the relation (38).

The terminology one uses for $\widehat{\mathcal{F}}$, comes obviously from the previous relations analogous to those obtained in the Maxwell theory. Indeed, since $(\widehat{\mathcal{G}}, \widehat{\mathcal{H}}) = 0$ because of the exactness condition in the sequence (41), one deduces:

$$\widehat{\mathcal{F}}_{[i\,j]} = \widehat{B}^k_{[\,i}(\partial_{|\,k\,|}\widehat{\mathcal{A}}_{j\,]}) + \widehat{\tau}^k_{[i,j]}\widehat{\mathcal{A}}_k,$$

and from the expression for $\widehat{\mathcal{H}}$:

$$\widehat{B}^h_{[\,k}(\partial_{|\,h}\widehat{\mathcal{F}}_{|\,j,i\,]}) + \widehat{\tau}^r_{[k,j}\widehat{\mathcal{F}}_{i,r]} = 0.$$

In the case of the weak fields limit, the latter become:

$$\begin{cases} \widehat{\mathcal{F}} & = & \mathbf{d}\widehat{\mathcal{A}} & \text{(Faraday tensor)} \\ \mathbf{d}\widehat{\mathcal{F}} & = & 0 & \text{(Bianchi identity).} \end{cases}$$

On the contrary, the symmetric part $\widehat{\mathcal{P}} \in S_2T^*\mathcal{M}$ of $\widehat{\mathcal{B}}$, called “the gravitational field tensor”, satisfies a first order PDE depending on $(\widehat{\mathcal{A}}, \widehat{\mathcal{B}})$, and thus $\widehat{\mathcal{P}}$ varies even if there is only an electromagnetic field! One has to notice that the symmetric part $(\partial_i\widehat{\mathcal{A}}_j)$ is never taken into account in physics, in contradistinction to $\widehat{\mathcal{A}}$. As a result all its derivatives should be physical observables. Finally, one deduces from the exactness of the complex (41) that no current of magnetic charges can exist. In other words, the Bianchi identity must be satisfied. Hence, the lack of magnetic charges can be justified in the framework of the Spencer cohomology of conformal Lie structures but not in the de Rham cohomology framework.

6. The dual linear Spencer complex and the Janet complex

6.1. The dual linear Spencer complex

One shall refer in this chapter to the definitions given in references (Goldschmidt 1976a, 1976b, 1978a, 1978b, 1981, Gasqui *et al.* 1984, Pommaret 1995) relating to the dual linear Spencer complex. Mainly, in this chapter, one has to build up the dual operator ${}^*\not{\mathbf{d}}'_{\mathbf{1}}$ of the infinitesimal operator $\not{\mathbf{d}}'_{\mathbf{1}}$ of $\not{\mathbf{D}}'_{\mathbf{1}}$, and the morphisms L_{n-i} $(i = 0, 1)$ in the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \theta_{\mathcal{M}} & \xrightarrow{j_1} & \underline{J_1} & \xrightarrow{\not{\mathbf{d}}'_{\mathbf{1}}} & \not{\mathbf{d}}'_{\mathbf{1}}(\underline{J_1}) \subseteq \not{\mathcal{C}}^{\mathbf{1}}_{\mathbf{1}} & \longrightarrow & 0 \\ & & & & \Big\downarrow{\scriptstyle L_n} & & \Big\downarrow{\scriptstyle L_{n-1}} & & \\ & & & & \not{\mathbf{A}}^{\mathbf{n}}_{\mathbf{1}} & \xleftarrow{{}^*\not{\mathbf{d}}'_{\mathbf{1}}} & \not{\mathbf{A}}^{\mathbf{n-1}}_{\mathbf{1}} & \longleftarrow & 0 \end{array}$$

where the $\not{\mathbf{A}}^{\mathbf{n-i}}_{\mathbf{1}}$ are the dualizing fiber bundles ($\not{\mathcal{C}}^{\mathbf{0}}_{\mathbf{1}} \equiv \underline{J_1}$):

$$\not{\mathbf{A}}^{\mathbf{n-i}}_{\mathbf{1}} = \underline{\Lambda^n\,T^*\mathcal{M}}\otimes_{\theta}{}^*\not{\mathcal{C}}^{\mathbf{i}}_{\mathbf{1}}.$$

One uses a rather classical method to determine the adjoint operators on a connected and oriented compact without boundaries (see P. J. Olver for instance 1986). To this purpose, one gets before a conformally equivariant Lagrangian density $\mathfrak{L}$:

$$\mathfrak{L} : \underline{J_1} \times_{\mathcal{M}} \not{\mathcal{C}}^{\mathbf{1}}_{\mathbf{1}} \equiv \not{\mathcal{C}}^{\mathbf{0}}_{\mathbf{1}} \times_{\mathcal{M}} \not{\mathcal{C}}^{\mathbf{1}}_{\mathbf{1}} \longrightarrow \underline{\Lambda^n\, T^*\mathcal{M}},$$

and one defines the morphisms L_{n-i} and ${}^*\not{\mathbf{d}}'_{\mathbf{1}}$ integrating by parts the infinitesimal variation of $\mathfrak{L}$, with $\not{\mathbf{d}}'_{\mathbf{1}}$ defined by ($\alpha_1 \equiv (\alpha, \beta)$):

$$\left\{ \begin{array}{rcl} \widehat{\mathcal{A}} & = & \mathbf{d}\alpha - \beta \\ \widehat{\mathcal{B}} & = & \mathbf{d}\beta - \mu \end{array} \right\} \equiv \not{\mathbf{d}}'_{\mathbf{1}} \alpha_1$$

where $\forall\, X, Y \in \mathcal{C}^1(T\mathcal{M})$:

$$\mu(X, Y) = \frac{1}{2}\beta(\nabla_X Y + \nabla_Y X) - c_0\, \alpha\, \omega(X, Y).$$

Let us define in $\not{\mathbf{A}}^{\mathbf{n-1}}_{\mathbf{1}}$, the images of $(\widehat{\mathcal{A}}, \widehat{\mathcal{B}})$ by the morphism L_{n-1} with the relations:

$$\left\{ \begin{array}{rcl} \widehat{\mathcal{J}} & = & (\partial\mathfrak{L}/\partial\widehat{\mathcal{A}}) \quad \text{(electric current)} \\ \\ \widehat{\mathcal{N}} & = & (\partial\mathfrak{L}/\partial\widehat{\mathcal{B}}), \end{array} \right.$$

and in $\not{\mathbf{A}}^{\mathbf{n}}_{\mathbf{1}}$, the images of α_1 by the morphism L_n with the relations:

$$\left\{ \begin{array}{rcl} \widehat{\mathcal{S}} & = & (\partial\mathfrak{L}/\partial\alpha) \quad \text{(density of entropy)} \\ \\ \widehat{\mathcal{Q}} & = & (\partial\mathfrak{L}/\partial\beta). \end{array} \right.$$

One obtains from the infinitesimal variation $\delta\mathfrak{L}$, the following defining relations of ${}^*\not{\mathbf{d}}'_{\mathbf{1}}$:

$$\left\{ \begin{array}{rcl} \widehat{\mathcal{S}} & = & \mathrm{div}\widehat{\mathcal{J}} - c_0 < \omega | \widehat{\mathcal{N}} > \\ \\ \widehat{\mathcal{Q}} & = & \widehat{\mathcal{J}} + \mathrm{div}_2 \widehat{\mathcal{N}} + < \zeta | \widehat{\mathcal{N}} >, \end{array} \right.$$

where div_2 and ζ are morphisms such as:

$$\begin{array}{rccl} \mathrm{div}_2 : & \overset{2}{\otimes}\, T\mathcal{M} & \longrightarrow & T\mathcal{M} \\ & u \otimes v & \longrightarrow & v\mathrm{div}(u) - u\mathrm{div}(v) + [u, v] \end{array}$$

and

$$\begin{array}{rccl} \zeta : & \overset{2}{\otimes}\, \mathcal{C}^1(T\mathcal{M}) & \longrightarrow & \mathcal{C}^0(T\mathcal{M}) \\ & u \otimes v & \longrightarrow & \zeta(u \otimes v) = \gamma(u)v = \gamma(v)u \equiv < \xi \mid u \otimes\, v > . \end{array}$$

From these results, a further step would be to know what the relations between ${}^*\not{\mathbf{d}}'_{\mathbf{1}}$ and the thermodynamical laws for the irreversible processes are, since the Lagrangian density $\mathfrak{L}$ depends on temperature and variables of internal states. Hence, $\mathfrak{L}$ might be identified with a Gibbs free enthalpy function, but moreover, setting certain conditions, the conformal equivariance laid down to $\mathfrak{L}$, might also allow us to consider $\mathfrak{L}$ as a wave-function, as we shall see in what follows.

6.2. The Janet complex of the Lagrangian density $\mathfrak{L}$

Again, one refers for the definitions to Pommaret's previous papers (Pommaret 1989, 1994) about the Janet complex and also to Gasqui-Goldschmidt's ones (Gasqui *et al.* 1988), but with an "un-named complex". The conformal equivariance of $\mathfrak{L}$ will merely allow us to obtain explicitly the first Janet operator $\mathbf{D}_1 : \mathcal{F}_0 \longrightarrow \mathcal{F}_1$ where $\mathcal{F}_0$ stands for the line bundle over $\mathcal{C}_{\mathbf{1}}^{\mathbf{0}} \times_{\mathcal{M}} \mathcal{C}_{\mathbf{1}}^{\mathbf{1}}$:

$$\begin{array}{c} \mathcal{F}_0 \\ \downarrow \uparrow \mathfrak{L} \\ \mathcal{C}_{\mathbf{1}}^{\mathbf{0}} \times_{\mathcal{M}} \mathcal{C}_{\mathbf{1}}^{\mathbf{1}} \end{array}$$

From now on, one presents a dependent coordinates formulation choosing an orthonormal system of local coordinates. To make $\mathbf{D}_1$, one must find the transformation rule of $\mathfrak{L}$ by the action of the conformal Lie pseudogroup. Namely, if $\hat{f}_2 \in \widehat{\mathcal{R}}_2$ and $\hat{f}_2^{-1} = \hat{g}_2$, one has the transformation rules:

$$\left\{ \begin{array}{rcl} y & = & \hat{f}(x) \simeq x + \hat{\xi}(x) \\ \alpha' \circ \hat{f} & = & \alpha - \frac{1}{n} \ln |\det J(\hat{f}_1)| \\ \beta_i' \circ \hat{f} & = & \hat{g}_i^j \circ \hat{f} \, [\beta_j - \frac{1}{n} \hat{g}_l^k \circ \hat{f} \, \hat{f}_{k\,j}^l] \\ \widehat{\mathcal{A}}_i' \circ \hat{f} & = & \hat{g}_i^j \circ \hat{f} \, \widehat{\mathcal{A}}_j \\ \widehat{\mathcal{B}}_{i,j}' \circ \hat{f} & = & \hat{g}_i^r \circ \hat{f} \; \hat{g}_j^s \circ \hat{f} \, \widehat{\mathcal{B}}_{r,s} \end{array} \right.$$

and one must have:

$$\mathfrak{L}(\hat{f}, \alpha' \circ \hat{f}, \beta' \circ \hat{f}, \widehat{\mathcal{A}}' \circ \hat{f}, \widehat{\mathcal{B}}' \circ \hat{f}) \det J(\hat{f}_1) = \mathfrak{L}(x, \alpha, \beta, \widehat{\mathcal{A}}, \widehat{\mathcal{B}}). \tag{45}$$

One obtains the infinitesimal condition and the definition of $\mathbf{D}_1$:

$$\mathbf{D}_1(\mathfrak{L}) \equiv \{ v^\mu(\mathfrak{L}) + (\operatorname{div} \xi^\mu) \, \mathfrak{L} \equiv \mathfrak{K} \} = 0,$$

whatever is v^μ ($\mu = 1, ..., \dim \mathfrak{g}_c$), generator of the Lie algebra $\mathfrak{g}_c$ of the conformal Lie group:

$$v^\mu = \hat{\xi}^{\mu,j} \partial_j + \phi^\mu(\alpha) \partial_\alpha + \phi^\mu(\beta_j) \partial_{\beta_j} + \phi^\mu(\widehat{\mathcal{A}}_j) \partial_{\widehat{\mathcal{A}}_j} + \phi^\mu(\widehat{\mathcal{B}}_{k,l}) \partial_{\widehat{\mathcal{B}}_{k,l}},$$

where

$$\left\{\begin{array}{rcl}\phi^{\mu}(\alpha) & = & -\frac{1}{n}\hat{\xi}^{\mu,k}_{k} \\ \phi^{\mu}(\beta_j) & = & -[\hat{\xi}^{\mu,k}_{j}\beta_k + \hat{\xi}^{\mu,k}_{k\,j}] \\ \phi^{\mu}(\widehat{\mathcal{A}}_j) & = & -\hat{\xi}^{\mu,k}_{j}\widehat{\mathcal{A}}_k \\ \phi^{\mu}(\widehat{\mathcal{B}}_{k,l}) & = & -[\hat{\xi}^{\mu,h}_{k}\widehat{\mathcal{B}}_{h,l} + \hat{\xi}^{\mu,h}_{l}\widehat{\mathcal{B}}_{k,h}]\end{array}\right.$$

and $\hat{\xi}^{\mu}_{2} \in \widehat{R}_2$. Moreover, $\mathfrak{K} \equiv (\mathfrak{K}^{\mu})$ is transformed like $\mathfrak{L}$ and therefore the zero section of $\mathcal{F}_1$ is defined with $\dim \mathfrak{g}_c = \dim \mathcal{F}_1$ constants c^{μ} such that $\mathfrak{K}^{\mu} = c^{\mu}\mathfrak{L}$ meaning that (45) (or the Lagrangian density) is defined up to a multiplicative constant. The second Janet operator $\mathbb{D}_2$ such as $\mathbb{D}_2 \circ \mathbb{D}_1 = 0$ and $\mathbb{D}_2 : \mathcal{F}_1 \longrightarrow \mathcal{F}_2$, is defined from the involution of the generators of $\mathfrak{g}_c$:

$$[v^{\mu}, v^{\nu}] = c^{\mu\,\nu}{}_{\lambda}\, v^{\lambda},$$

where the $c^{\mu\,\nu}{}_{\lambda}$ are the constants of structure of the $\mathfrak{g}_c$ (see P. J. Olver 1986 for a precise definition of the bracket used above). Thus, one obtains:

$$\mathbb{D}_2(\mathfrak{K}) \equiv \left\{v^{\mu}(\mathfrak{K}^{\nu}) - v^{\nu}(\mathfrak{K}^{\mu}) - c^{\mu\,\nu}{}_{\lambda}\, \mathfrak{K}^{\lambda} + (\operatorname{div}\hat{\xi}^{\mu})\, \mathfrak{K}^{\nu} - (\operatorname{div}\hat{\xi}^{\nu})\, \mathfrak{K}^{\mu}\right\}.$$

Leading down $\mathfrak{K}^{\mu} = c^{\mu}\mathfrak{L}$ in the latter expression, one deduces the following constraints on the constants $c^{\mu\,\nu}{}_{\lambda}$ of structure:

$$c^{\mu\,\nu}{}_{\lambda}c^{\lambda} = 0,$$

with $\mathbb{D}_1(\mathfrak{L})^{\mu} = c^{\mu}\mathfrak{L}$.

This latter equation gives in fact only one arbitrary constant among the c^{μ}'s and perhaps might be ascribed to a constant such as the Planck one.

As a particular case, if the metric ω and $\widehat{\mathcal{A}}$ are constants, α and β vanish and $\mathfrak{L}$ function of x and $\widehat{\mathcal{A}}$ only, such that there exists η satifying the relation

$$\widehat{\mathcal{J}}^{k} = \eta^{k}\mathfrak{L},$$

then $\forall\,\mu, h$ one has $\hat{\xi}^{\mu,k}_{k} = \hat{\xi}^{\mu,k}_{k\,h} = 0$ and

$$\left[\hat{\xi}^{\mu,k}\partial_k - (\eta^{k}\hat{\xi}^{\mu,h}_{k})\widehat{\mathcal{A}}_h\right] \mathfrak{L} - c^{\mu}\, \mathfrak{L} = 0.$$

Thus, one obtains an analogous PDE to the Dirac equation but being equivariant and not only covariant like the Dirac equation. Nevertheless $\mathfrak{L}$ being a real function, it can be interpreted as a wave-function only if one has a definition of a measure of probability. We propose two ways of doing this in the conclusion, and we make suggestions for taking the weak and strong interactions into account in the model.

7. Conclusion

The function $\mathfrak{L}$ being real, let us suggest a notion of mesure of probability. Two of them can be proposed. The first one is related to an "à la Misra-Prigogine-Courbage" (MPC) (Misra *et al.* 1979, Misra 1987) approach. With more details, the wave-function $\mathfrak{S}$ of the model would be the function obtained when applying the integral action of the evolution operator to an initial Lagrangian density $\mathfrak{L}$. First, we thus obtain a wave interpretation. Second, within the MPC theory, the reduction of the wave-paquet (or the collapse) would axiomatically be defined as the achievement of a K-partition of functions $\mathfrak{S}$ on a compact (on which the initial Cauchy data would be defined) of a submanifold of the Minkowski space-time to determine a single function $\mathfrak{S}$. This K-partition of functions $\mathfrak{S}$ would then be a set of new initial Lagrangian densities $\mathfrak{L}$ before any time evolution. The physical measurement thus achieved would not lead us to obtain pure states and with the assumption (to be confirmed) that the conformally equivariant Lagrangian densities $\mathfrak{L}$ would be Kolmogorov flows, it would then be possible, according to the MPC theory, to build up a non-commutative algebra of observables on the analogy of quantum mechanics. Moreover, the K-partition would be achieved on a set of Lagrangian densities of type (m,s) deduced by projection on the basis of the eigenstates of the Poincaré group during the process of measurement (from a certain point of view, the apparatus of measurement, being themselves of the type (m,s)). Thus, as a result of this projection, we shall see a particle state and initial Lagrangian densities of type (m,s) not depending on the fields of interaction anymore. The particle states might be interpreted as a split up of variables between the variable of position x and the fields variables (during specific physical measurements). In other words, one could say that the energy of the fields of interaction would be transfered by radiation to the apparatus of measurement. From a more general point of view, the process of "fragmentation" would be defined by the change from a tensorial Hilbert space to a decomposable Hilbert space ($\mathcal{H}_i$; $i = 1, 2$: Hilbert spaces):

$$\mathcal{H}_1 \otimes \mathcal{H}_2 \longrightarrow \mathcal{H}_1 \oplus \mathcal{H}_2$$

Process that might be interpreted as well as a separation of phase in thermodynamics for the function $\mathfrak{L}$ is similar to a Gibbs function. Under these conditions, the analytical developments of the initial Lagrangian densities define tensors of susceptibility directly associated with the "classical" states of the condensed matter, therefore with the classical notion of macroscopic particles. Thus, the macroscopicity would be associated with the "degree" of separability of the initial densities $\mathfrak{L}$. From our point of view, a second possibility used to define a measure of probability would be to refer to the Penrose approach (Penrose *et al.* 1986) on the complexification of the wave-function in the frame of the theory of twistors, with moreover, the notion of measure of probability defined as it usually is in the first quantization. This procedure is completly different from the one consisting in simply making an algebraic extension from the field of reals to the field of complexes, of the wave-functions satisfying PDE on the field of reals. Such an extension would lead to complex solutions whose real and imaginary parts would each separatly be solutions. Consequently, we assume that a complexification presenting a real interest would give access to the determination of complex solutions which, on the contrary, would be nei-

ther their real parts, nor their imaginary ones, solutions for the given system of PDE. To achieve that, Penrose suggested decomposing a problem defined on the real Minkowski space $\mathcal{M}$ into a set of physical subsystems defined on submanifolds of $\mathcal{M}$ and having at least one (complex) spin structure, like the celestial spheres $S^{\pm}$ are, or at fixed time, the sphere S^3. The spin structures are associated with vector bundles like the tangent spaces, or on a rather similar manner (up to the target manifold), the 1-jet bundle $J_1(\epsilon)$ for example. In the present situation, we shall refer to the $\not{\mathbf{d}}'_{\mathbf{1}}(\underline{J_1}) \subseteq \not{\mathscr{C}}^{\mathbf{1}}_{\mathbf{1}}$ bundle over $\underline{J_1}$. We therefore must present the tensors $\widehat{\mathcal{A}}$ and $\widehat{\mathcal{B}}$ with spinors. In the more simple case with only one spin structure (on S^2 for example) and taking up again the Penrose indexed notation, one need at least 3 spinors to decompose the tensor $\widehat{\mathcal{A}}$:

$$\widehat{\mathcal{A}}_a = \alpha_A.\overline{\alpha}_{A'} + \beta_A.\overline{\beta}_{A'} - \gamma_A.\overline{\gamma}_{A'}.$$

For only under these conditions, $\widehat{\mathcal{A}}$ is of any norm, but with a spinor, $\widehat{\mathcal{A}}$ must be of the type light and with two spinors, the norm of $\widehat{\mathcal{A}}$ must be non-negative or non-positive. As far as I know this simple decomposition has never been presented yet (apart from the one with 2 spinors or 3 maximum and with a + mark). But it has indeed a few avantages. First, it is associated (Bars *et al.* 1990) with the representations $\{3\}$ and $\{\bar{3}\}$ of $SU(2,1)$ isomorphic to the one of $SU(3)$. It is only about representations; the group $SU(2,1)$ not being a dynamical gauge group of the model, like for example the Yang-Mills type, but only a group of internal classification and of invariance of the decomposition. The non-compactness of the group cannot appear determining to us within this context. But its finished unitary irreducible representations constructed from the two fundamental representations $\{3\}$ and $\{\bar{3}\}$ only do. Moreover, a symmetry breaking from $SU(2,1)$ to $SU(2)\times U(1)$ or $SU(1,1)\times U(1)$ is associated to a symmetry breaking of type T (for instance during the process of fragmentation) and $\widehat{\mathcal{A}}$ is then of a non-negative or non-positive norm. All of that suggests a possible link with a theory of weak and strong interactions. Before that, formally deriving the spinors decomposing $\widehat{\mathcal{A}}$, we obtain the following decomposition for $\widehat{\mathcal{B}}$ (Penrose *et al.* 1986, see chapter 4.4.7.) using the Leibniz law for the spinors:

$$\widehat{\mathcal{B}}_{a,b} = \alpha_A.\overline{\Gamma}_{B'A',B} + \overline{\alpha}_{A'}.\Gamma_{BA,B'} + \beta_A.\overline{\Theta}_{B'A',B} + \overline{\beta}_{A'}.\Theta_{BA,B'} + \gamma_A.\overline{\Omega}_{B'A',B} + \overline{\gamma}_{A'}.\Omega_{BA,B'},$$

where Γ, Θ and Ω are an any mixed spinors of valence (2,1). They can decompose themselves into irreducible spinors with respect to $SL(2,\mathbf{C})$:

$$\Gamma_{AB,B'} = \Gamma_{(AB),B'} - \frac{1}{2}\epsilon_{[AB]}\Gamma_{C\ \ ,B'}^{\ \ C}$$

(with similar expressions for Θ and Ω). Let us notice that at a given fixed time t and with (α,β) fixed (i.e. during the measure), $SL(2,\mathbf{C})$ being associated with $O^{+}_{\uparrow}(1,3)$, the irreducibility is thus, according to $SU(2)$, locally with respect to $O(3)$. Moreover, $\Gamma_{(AB),B'}$ is defined by 6 complex components, linearly independant and $\Gamma_{C\ \ ,B'}^{\ \ C}$ by 2 complex components. The interest of such a decomposition appears as soon as the assumption is made that the Lagrangian density $\mathfrak{L}$ is holomorphic in the various spinors on the

submanifolds endowed with spin structures. Then $\mathfrak{L}$ satisfies complex PDE deduced from the real PDE defined on $\mathcal{M}$ by lifting on the spinors bundle. But, the physical meaning of the holomorphy is that the physical system can precisely be confined on the submanifolds of M endowed with a spin structure. If we make the physical interpretation that to the spinors decomposing $\widehat{\mathcal{A}}$ and $\widehat{\mathcal{B}}$ are associated some fields of interactions or particles, it means that the latter can only be considered as "free" if they are precisely confined on those submanifolds; the density $\mathfrak{L}$ being then a complex solution on the spinors bundle of the various lifted PDE of the model, from which neither the real part, nor the imaginary part are any solutions. On that subject, one cannot help thinking of the very controversial - and to my knowledge still un-confirmed - Larue *et al.* (1977) and Schaad *et al.* (1981) experiences on "free" quarks confined on bidimensional ($\simeq S^2$) layers of superconducting (!) nobium covering microballs of tungsten (see also Goldman 1995, 1996). Especially because the decomposition presented above could suggest that the simple spinors α, β and γ would be associated with some fields of fermion of mono-colored gluons of spin 1/2 (and not to bi-colored bosons!) and that the symmetric mixed spinors of valence (2,1) would be 3 quarks states of spin 1/2 determined by the 3 contracted spinors $\Gamma_C{}^C{}_{,B'}$, $\Theta_C{}^C{}_{,B'}$ and $\Omega_C{}^C{}_{,B'}$ on which $SU(2,1)$ acts. At last, to finish as well as to come back to an application of the physics of the superconducting states (that some of the theoretical physicists like Mendelstam (1982) or t'Hooft (1978) used to explain the quarks confinement) the question is under which conditions the 4-vector current:

$$\widehat{\mathcal{J}} \;=\; (\partial\mathfrak{L}/\partial\widehat{\mathcal{A}})$$

would be anomalous as well as the Faraday tensor $\widehat{\mathcal{F}}$. A possibility we suggest would be to consider a third metric λ deduced from ν like ν is from ω. The latter metric λ would appear in matter such as in crystals or amorphous materials for instance. Indeed in this case, the second metric ν (and its corresponding potentials and fields) would be associated to the "crystal field" and to a new kind of substrat space-time and new specific succeptibilities. But in the contrary to ω the Weyl curvature associated to ν is no longer necessarly vanishing as we pointed out in a previous chapter. Thus, the corresponding Lie equations for conformal transformations won't be involutive. As a result, the corresponding Spencer sequences and the relative one won't be exact any more as well. In particularly magnetic charges might occur and so anyons (Wilczek 1990). In the framework of symplectic cohomologies of the Lagrangian density $\mathfrak{L}$, it can be shown that the anomalies classification is associated to a second cohomology space knowledge as well as a third one (Cariñena *et al.* 1988). The question remaining to work out would be to know if one can obtain integer cohomologies from these latter in order to exhibit a kind of "quantum structure". To conclude on a more philosophical note, the wave-function model we present here could be interpreted within the framework of a non-Lavoisian chemistry like G. Bachelard (1940) developped. In actual fact, the quantons would not be of "substance", but as Bachelard named them, they would rather be "grains de réaction" and thus, from a certain point of view, like quantons of reactional synthesis or of separation of phase associated with the concept of fragmentation. We also think that this model can't describe global evolution such as a big-bang model but only a local

evolution. In some ways as G. Deleuze *et al.* (1980) say about their striated space (like a substrat space-time) and smooth space concepts (see chapters: "12.1227. Traité de nomadologie: la machine de guerre" and "11.1837. De la ritournelle"), "one can only know the path by exploring it". Also, the form concept for matter in space-time could be related to a boundary being the geometrical set of places onto which the "grains de réaction" occur.

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References

Asher, E. 1973 In *Magnetoelectric Interaction Phenomena in Crystals.* Proceedings, Seattle, 69.

Bachelard, G. 1940 *La philosophie du non.* Paris: Presse Universitaire de France.

Bacry, H. 1967 *Leçons sur la théorie des groupes et les symétries des particules élémentaires.* Paris: Gordon and Breach.

Bars, I. & Zhong-jian, T. 1990 *The unitary irreducible representations of SU(2,1). J. Math. Phys.* (7)**31**, 1576-1586.

Bateman, H. 1910 *Proc. Lond. math. Soc.* **8**, 223.

Cariñena, J. F. & Ibort, L. A. 1988 *J. Math. Phys.* **29**, 541.

Cunningham, E. 1910 *Proc. Lond. math. Soc.* **8**, 77.

Deleuze, G. & Guattari, F. 1980 *Mille Plateaux.* Paris: Les éditions de minuit.

Dieudonné, J. 1971 *Éléments d'analyse, Tome IV.* Paris: Gauthier-Villars Éditeur.

Gasqui, J. & Goldschmidt, H. 1988 *Complexes of Differential Operators and Symmetric Spaces.* In *Deformation Theory of Algebras and Structures and Applications.* 797-827 (eds. M. Hazewinkel & M. Gerstanhaber) Dordrecht-Boston-London: Kluwer Academic Publishers.

Gasqui, J. & Goldschmidt, H. 1984 *Deformations infinitesimales des structures conformes plates. Progress in Mathematics*, Vol. **52**, Boston-Basel-Stuttgart: Birkhaeuser.

Goldman, V. J. 1996 *Surf. Sci.* **361**, 1.

Goldman, V. J. & Su, B. 1995 *Science* **267**, 1010.

Goldschmidt, H. & Spencer, D. 1976a *On the non-linear cohomology of Lie equations. Part I, Acta Math.* **136**, 103-170.

Goldschmidt, H. & Spencer, D. 1976b *On the non-linear cohomology of Lie equations. Part II, Acta math.* **136**, 171-239.

Goldschmidt, H. & Spencer, D. 1978a *On the non-linear cohomology of Lie equations. Part III, J. Differ. Geom.* **13**, 409-453.

Goldschmidt, H. & Spencer, D. 1978b *On the non-linear cohomology of Lie equations. Part IV, J. Differ. Geom.* **13**, 455-526.

Goldschmidt, H. 1981 *On the nonlinear cohomology of Lie equations. Part V, J. Differ. Geom.* **16**, 595-674.

't Hooft, G. 1978 *Nucl. Phys.* B**138**, 1.

Itzykson, C. & Zuber, J.-B. 1980 *Quantum Field Theory.* McGraw-Hill.

Janner, A. & Asher, E. 1969 *Phys. Lett.* (4)A**30**, 223.

Janner, A. & Asher, E. 1978 *Physica* **48**, 425.

Katanaev, M. O. & Volovich, I. V. 1992 *Ann. Phys.* **216**, 1-28.

Kleinert, H. 1989 *Gauge fields in condensed matter.* Vol. **II**. World Scientific.

Kumpera, A. & Spencer, D. C. 1972 *Lie Equations, Vol. I: General Theory. Annals of Mathematics Studies*, Vol. **73**.

Larue, G. S.& Fairbank, W. M. & Hebard, A. F. 1977 *Evidence for the Existence of Fractional Charge on Matter*. Phys. Rev. Lett.* (18)**38**, 1011-1014.

Lévy-Leblond, J.-M. 1990 *ENIGMA OF THE SPħINx.* In *Symposium on foundations of modern Physic.* 226-241. World Scientific.

Lurçat, F. 1964 *Quantum Field Theory And The Dynamical Role Of Spin. Physics* (2)**1**, 95-106.

Mandelstam, S. 1982 *Monopoles in Quantum FIeld Theory.* In *Proceedings of the Monopole Meeting.* Trieste, Italy, December 1981. World Scientific.

Misra, B. & Prigogine, I. & Courbage, M. 1979 *From deterministic dynamics to probabilistic descriptions. Physica* A**98**, 1-26.

Misra, B. 1987 *Fields as Kolmogorov Flows. J. Stat. Phys.* (5/6)**48**, 1295-1320.

Olver, P. J. 1986 *Applications of Lie Groups to Differential Equations. Graduate texts in mathematics*, Vol. **107**. Springer.

Penrose, R. & Rindler, W. 1986 *Spinors and Space-Time. Cambridge Monographs On Mathematical Physics*, Vol. **I** & **II**. Cambridge University Press.

Pommaret, J.-F. 1994 *Partial Differential Equations and Group Theory, New Perspectives for Applications. Mathematics and Its Applications*, vol. **293**. Dordrecht-Boston-London: Kluwer Academic Publishers.

Pommaret, J.-F. 1995 *Suites Differentielles et Calcul Variationnel. C. R. Acad. Sci. Paris* **320**, Série I, 207-212.

Pommaret, J.-F. 1989 *Gauge Theory And General Relativity. Reports in Mathematical Physics* (3)**27**, 313-344.

Pommaret, J.-F. 1988 *Thermodynamique et théorie des groupes. C. R. Acad. Sci. Paris* **307**, Série I, 839-842.

Rubin, J. L. 1994 *Relativistic Crystalline Symmetry Breaking And Anyonic States in Magnetoelectric Superconductors. Ferroelectrics* **161**, 335.

Rubin, J. L. 1993 *Broken Relativistic Symmetry Groups, Toroidal Moments and Superconductivity In Magnetoelectric Crystals. Il Nuovo Cimento D-Cond. Matt. At.* **15**, 59.

Schaad, L. J. & Hess Jr., B. A. & Wikswo Jr., J. P. & Fairbank, W. M. 1981 *Quark chemistry. Phys. Rev.* (4)A**23**, 1600-1607.

Shih, W. 1986 *Une méthode élémentaire pour létude des équations aux dérivées partielles.* In *Diagrammes*, **16**. Paris.

Shih, W. 1987 *C. R. Acad. Sci. Paris* **304**, Série I, 103-106, 187-190, 535-538.

Shih, W. 1991 *Gazette. Société Mathématique de France*, 1.

Spencer, D. C. 1962 *Deformation of Structures on Manifolds defined by transitive, continuous pseudogroups. Parts I & II, Annals of Math.* **76**, 306-445.

Spencer, D. C. 1965 *Deformation of Structures on Manifolds defined by transitive, continuous pseudogroups. Part III, Annals of Math.* **81**, 289-450.

Wilczek F. 1990 *Fractional Statistics and Anyons Superconductivity.* World Scientific Publishing Co. Pte. Ltd.